Designing new apartment buildings for strings and conformal field theories. First steps

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Abstract : The concepts of apartments and buildings were suggested by Tits for description of the Weyl-Coxeter reflection groups. We use these and many additional facts from the theory of reflection and pseudo-reflection groups along with results from the algebraic and symplectic geometry of toric varieties in order to obtain the tachyon-free Veneziano-like multiparticle scattering amplitudes and the partition function generating these amplitudes. Although the obtained amplitudes reproduce the tachyon-free spectra of both open and closed bosonic string, the generating (partition) function is not that of the traditional bosonic string. It is argued that it is directly related to the N=2 supersymmetric quantum mechanical model proposed by Witten in 1982 in connection with his development of the Morse theory. Such partition function can be independently obtained with help of the results by Solomon (published in 1963) on invariants of finite (pseudo) reflection groups. Although the formalism developed in this work is also applicable to conformal field theories (CFT), it leaves all CFT results unchanged.

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1 From geometric progression to Weyl character formula

1.1 General considerations

Consider finite geometric progression of the type

$$
\begin{aligned}
\mathcal{F}(c,n) &= \sum_{l=-n}^{n}\exp\{cl\} = \exp\{-cn\}\sum_{l=0}^{\infty}\exp\{cl\} + \exp\{cn\}\sum_{l=-\infty}^{0}\exp\{cl\} \\
&= \exp\{-cn\}\frac{1}{1-\exp\{c\}} + \exp\{cn\}\frac{1}{1-\exp\{-c\}} \\
&= \exp\{-cn\}\left[\frac{\exp\{c(2n+1)\}-1}{\exp\{c\}-1}\right]. \qquad (1.1)
\end{aligned}
$$

The reason for displaying the intermediate steps will be explained shortly. But first, we would like to consider the limit : $c \to 0^+$ of $\mathcal{F}(c,n)$. Clearly, it is given by $\mathcal{F}(0,n) = 2n+1$. The number $2n+1$ equals to the number of integer points in the segment $[-n,n]$ *including boundary* points. It is convenient to rewrite the above result in terms of $x = \exp\{c\}$. So that we shall write formally $\mathcal{F}(x,n)$ instead of $\mathcal{F}(c,n)$. Using such notations, let us consider the related function[2]

$$\bar{\mathcal{F}}(x,n) = (-1)\mathcal{F}(\frac{1}{x},-n). \qquad (1.2)$$

Explicitly, we obtain:

$$\bar{\mathcal{F}}(x,n) = (-1)\frac{x^{-(-2n+1)}-1}{x^{-1}-1}x^{n}. \qquad (1.3)$$

In the limit: $x \to 1+0^+$ we obtain $\bar{\mathcal{F}}(1,n) = 2n-1$. The number $2n-1$ is equal to the number of integer points *inside* the segment $[-n,n]$.

These seemingly trivial results will be broadly generalized in this work. To this purpose let us introduce some notations. First, we replace x^l by $\mathbf{x}^{\mathbf{l}} = x_1^{l_1}x_2^{l_2}\cdots x_d^{l_d}$ and, accordingly, we replace the summation sign in the left hand side of Eq.(1.1) by the multiple summation, etc. Thus obtained function $\mathcal{F}(\mathbf{x},n)$ in the limit $x_i \to 1+0^+$, $i=1-d$, produces the anticipated result:

$$\mathcal{F}(\mathbf{1},n) = (2n+1)^d. \qquad (1.4)$$

It provides the number of integer points *inside* the $d-$dimensional cube and at its faces. Analogously, using Eq.(1.2), we obtain

$$\bar{\mathcal{F}}(\mathbf{1},n) = (2n-1)^d. \qquad (1.5)$$

[2]Incidentally, such type of relation (the Erhart-Macdonald reciprocity law) is characteristic for the Ehrhart polynomial for rational polytopes. Work of Stanley, Ref.[1], provides many applications of this law.

This is just the number of integer points *strictly inside* of d-dimensional cube. In our earlier work [2] Eq.(1.5) (with $2n$ being replaced by N and d being replaced by $n+1$) was obtained and interpreted as the Milnor number of the Brieskorn-Pham (B-P) type singularity, e.g. that of the Fermat-type. The above number counts "new" Veneziano-like amplitudes which mathematically are interpreted as periods of the Fermat-type hypersurfaces. The purpose of this work is to make such identification more accurate. It becomes possible based on detailed Lie group-theoretic analysis of earlier obtained "new" amplitudes. As in known treatments of conformal field theories (CFT) whose computational formalism and its physical interpretation comes directly from that developed in string theories [3], we demonstrate that the same is also true in the present case. The same group-theoretic methods which work for the Veneziano-like amplitudes can be extended to recover CFT results. However, the physical models leading to these "new " amplitudes are noticeably different from those used traditionally.

This work is made to a large degree self contained and is organized in a such way that it is expected to be accessible to both mathematicians and physicists. Such attitude towards our readers had made this paper somewhat larger than usual. At the same time, in writing this paper we wanted to expose new elements with sufficient details which can be easily checked and understood.

In particular, already the rest of Section 1 sets up the tone for the reminder of this paper. In this section we consider various aspects of the Weyl character formula from nontraditional point of view. These include: group-theoretic, combinatorial and dynamical. Some results of this section are new and as such have independent interest but most of the results are auxiliary and used in the rest of the paper. Section 2 contains some facts from the theory of linear algebraic groups. These facts are intertwined with the description of affine and projective toric varieties. Although some readers might be familiar with such objects our exposition differs from more traditional and highly recommended mathematical expositions, e.g. [4], because it contains various physical applications. Section 3 is built on the results of section 2 and contains some discussion of the method of coadjoint orbits and moment map to be used later in Section 5. Although such topics are usually discussed from the point of view of symplectic geometry [5], the results of Section 2 allow us to look at these topics from group-theoretic perspective only. Section 3 also contains few new results and is mainly auxiliary. The new results appear first in Section 4 devoted to detailed study of the Veneziano-like amplitudes. If for the reading of Sections 1-3 use of the results of Appendix A is helpful, the reading of Section 4 is closely interconnected with the results of the Appendix B in which we discuss the analytical properties of the Veneziano and Veneziano-like amplitudes. The difference between the Veneziano and Veneziano-like amplitudes lies in the fact that the last ones are tachyon-free by design. Mathematically, these amplitudes are periods associated with Hodge-De Rham cohomology basis made of differential forms living on Fermat-type hypersurfaces. Recently published monograph [6] provides an easily readable comprehensive mathematical exposition of results related to periods associated with complex hypersurfaces. Many physically relevant applications of these mathematical results can be found in our earlier work,

Ref.[2]. Section 4 contains also some discussion relating the Veneziano-like amplitudes to the amplitudes in CFT. Roughly speaking, the differences between these observables are very much the same as the differences between the calculable observables in periodic solids and those in the vacuum in the absence of periodicity. Such differences are known and treated accordingly in the theoretical solid state physics. Effectively, CFT formalism accounts for the periodicity effects. Deeper reasons revealing the true essence of the affine Kac-Moody algebras are provided in the Appendix A from which it should be also clear how such results can be extended if necessary. Section 4 contains as well some solutions to exercises (Chapter5, paragraph 5, problem set # 3) from the book by Bourbaki on Lie groups and Lie algebras [7] while Section 5 contains the rest of solutions to these exercises. These exercises are based (in part only !) on the fundamental paper by Solomon [8] on invariants of (pseudo) reflection groups published in 1963. The results by Solomon allow us to reanalyze group-theoretically the Veneziano-like amplitudes and to reconstruct the partition function reproducing these amplitudes. This is accomplished in Section 5 where we discuss 3 independent ways : group-theoretic, symplectic and supersymmetric -all leading to the same partition function for the Veneziano-like amplitudes. We argue in this section that the finite dimensional N=2 supersymmetric quantum mechanical model proposed by Witten in 1982 is sufficient for reproduction of the Veneziano-like amplitudes and the associated with them partition function. Nevertheless, using some group-theoretic considerations inspired by book by Ginzburg [9] and taking into account the Lefschetz isomorphism theorem [10], we map Witten's supersymmetric quantum mechanical model into the direct sum of $sl_2(\mathbf{C})$ Lie algebras. Ultimately, this allows us to reformulate the associated quantum mechanical problems in the language of semisimple Lie algebras (since all of them are made of copies of $sl_2(\mathbf{C})$ [11]). Based on the book by Kac [12], this allows us to find very precisely the place at which the existing formalism of CFT and the formalism of our new "string" theory come apart. Naturally, by sacrificing mathematical rigor, this allows, in principle, to recover back the results of "old" string theories. In Section 6 we briefly discuss some implications of the obtained results. These are related to remarkable connections between the pseudo-reflection groups used in the main text and complex hyperbolic geometry. Such connections tie together new string theory, hyperbolic space and quantum mechanics thus making them inseparable. This connection is rigid in the sense that variations of complex structure (that is "motions" in the moduli space) of complex projective hypersurfaces, e.g. of Fermat type, do not destroy such connections.

1.2 Connection with integral polytopes

Let $\mathbf{Z}^d$ be a subset of Euclidean space $\mathbf{R}^d$ consisting of points with integral coefficients. Select a subset $\mathcal{P} \subset \mathbf{R}^d$. A convex hull of the intersection $\mathcal{P} \cap \mathbf{Z}^d$ is called the *integral polytope*. The scalar product in Euclidean space $\mathbf{R}^d$ is

defined by

$$< \mathbf{x}, \mathbf{y} >= \sum_{i=1}^{d} x_i y_i. \tag{1.6}$$

With help of such definition the d-dimensional version of Eq.(1.1) is written below

$$\sum_{\mathbf{x}\in\mathcal{P}\cap\mathbf{Z}^d} \exp\{< \mathbf{c}, \mathbf{x} >\} = \sum_{\mathbf{v}\in Vert\mathcal{P}} \exp\{< \mathbf{c}, \mathbf{v} >\} \left[\prod_{i=1}^{d}(1-\exp\{-c_i u_i^v\})\right]^{-1}. \tag{1.7}$$

Here $Vert\mathcal{P}$ denotes the vertex set of the integral polytope, in our case the d-dimensional cube. It is made of the highest weight vectors of the Weyl-Coxeter reflection group B_d appropriate for cubic symmetry as explained in the Appendix A. The set $\{u_1^v, ..., u_d^v\}$ is made of vectors (not necessarily of unit length) constituting the orthonormal basis of the $d-$dimensional cube. These vectors are oriented along the positive semi axes with respect to center of symmetry of the cube. When parallel translated to the edges ending at particular hypercube vertex v, they can point either in or out of this vertex. The correctness of Eq.(1.7) can be readily checked by considering Eq.(1.1) as an example. Subsequent generalization of this result to a square, a cube, etc. causes no additional problems.

The above formula was obtained (seemingly independently) in many different contexts. For instance, in the context of discrete and computational geometry it is attributed to Brion [13]. In view of Eq.(1.2), it can as well to be attributed to Ehrhart [14] and to many others but, actually, this is just a special case of Weyl's character formula as we are going to demonstrate shortly below. To prove that this is indeed the case we need to recall some results of Atiyah and Bott [15] nicely summarized in Ref. [16] by Bott. It is physically illuminating, however, to make a small digression in order to discuss the dynamical transfer operators introduced by Ruelle [17]. This discussion will lead us directly to the results of Atiyah and Bott [15]. In addition, it enables us to look at these results from the point of view of thermodynamic formalism developed by Ruelle for description of evolution of chaotic dynamical systems. Such discussion is helpful for interpreting the Veneziano-like amplitudes in terms of dynamical systems formalism as suggested in our earlier work, Ref.[2].

1.3 From Ruelle dynamical transfer operator to Atiyah and Bott Lefschetz-type fixed point formula for elliptic complexes

To emphasize the physical content of Ruelle transfer operator we would like to reproduce several relevant results from the unpublished book Cvitanovič [18] [3]. Following [18], any classical dynamical system can be thought of as the

[3] Professor Cvitanović had kindly pointed to the author the web site from which the book was downloaded

pair $(\mathcal{M},\mathbf{f})$ with $\mathbf{f}$ being a map $\mathbf{f}$: $\mathcal{M}\rightarrow\mathcal{M}$. More specifically, if the initial conditions at time $t=0$ are given by the phase vector $\boldsymbol{\xi}=\mathbf{x}(0)$, then the mapping is just the phase space trajectory: $\mathbf{x}(t)$=$\mathbf{f}^t(\boldsymbol{\xi})\equiv\mathbf{f}^t(\mathbf{x}(\mathbf{0}))$. Let $\mathbf{o}=\mathbf{o}(\mathbf{x}(t))$ be some physically meaningful observable. We associate with it the integrated observable $\mathbf{O}^t(\boldsymbol{\xi})$ defined by

$$\mathbf{O}^t(\boldsymbol{\xi})=\int\limits_0^t d\tau\mathbf{o}(\mathbf{f}^\tau(\boldsymbol{\xi})). \tag{1.8}$$

In terms of these definitions the *evolution operator* $\mathcal{L}^t$ (the transfer operator) is formally defined as

$$\mathcal{L}^t(\mathbf{y},\mathbf{x})=\delta(\mathbf{y}-\mathbf{f}(\mathbf{x}))\exp(\beta\mathbf{O}^t(\mathbf{x})). \tag{1.9}$$

In the case when the observable $\mathbf{o}$ is classical Hamiltonian the parameter β is a sort of the inverse dynamical temperature. More rigorously, the operator $\mathcal{L}^t$ is defined by its action on the bounded scalar function $h(\mathbf{x})$ "living" on the phase space $\mathcal{M}$:

$$\mathcal{L}^t h(\mathbf{y})=\int\limits_{\mathcal{M}} d\mathbf{x}\delta(\mathbf{y}-\mathbf{f}(\mathbf{x}))\exp(\beta\mathbf{O}^t(\mathbf{x}))h(\mathbf{x}). \tag{1.10}$$

Such operator possess nice semigroup property

$$\mathcal{L}^{t_1+t_2}(\mathbf{y},\mathbf{x})=\int\limits_{\mathcal{M}} d\mathbf{z}\mathcal{L}^{t_2}(\mathbf{y},\mathbf{z})\mathcal{L}^{t_1}(\mathbf{z},\mathbf{x}) \tag{1.11}$$

which should look familiar to every physicist dealing with Feynman's path integrals. In this work we have no intentions to develop this avenue of thought however. Instead, assuming that the phase trajectory possesses fixed point(s), i.e. if we assume that equation $\mathbf{x}=\mathbf{f}^t(\mathbf{x})$ (or, more generally, $\mathbf{f}^n(\mathbf{x})=\mathbf{x}$ with $\mathbf{x}$ being the periodic point of period n) possesses at least one fixed point solution, the trace of the evolution operator (the *partition function* in physical language) can be written as follows

$$tr\mathcal{L}^n=\int\limits_{\mathcal{M}} d\mathbf{x}\mathcal{L}^n(\mathbf{x},\mathbf{x})=\sum_{\mathbf{x}_i\in Fix\mathbf{f}^n}\frac{\exp(\beta\mathbf{O}^t(\mathbf{x}_i))}{|\det(\mathbf{1}-\mathbf{J}^n(\mathbf{x}_i))|} \tag{1.12}$$

where the set $Fix\mathbf{f}^n$ is defined by $Fix\mathbf{f}^n=\{\mathbf{x}:\mathbf{f}^n(\mathbf{x})=\mathbf{x}\}$ and $\mathbf{J}^n(\mathbf{x}_i)$ denoting the Jacobian matrix to be defined more accurately shortly below. This result is useful to compare with Eq.(1.7). Clearly, when the denominator of the right hand side of Eq.(1.7) is rewritten in terms of determinants both expressions become identical.

To make connections with the work of Atiyah and Bott [15], we would like to rewrite just presented intuitive physical results in a more mathematically systematic way. Following Ruelle[17], let us consider a map f: $\mathcal{M}\rightarrow\mathcal{M}$ and

a scalar function g: $\mathcal{M} \to \mathbf{C}$.Based on these data, the transfer operator $\mathcal{L}$ is defined now by

$$\mathcal{L}\Phi(x) = \sum_{y:fy=x} g(y)\Phi(y). \tag{1.13}$$

By analogy with Eq.(1.11), if $\mathcal{L}_1$ and $\mathcal{L}_2$ are transfer operators associated with maps f_1 , f_2 : $\mathcal{M} \to \mathcal{M}$ and weights g_1 and g_2 such that, as before, $\mathcal{M} \to \mathbf{C}$ then,

$$(\mathcal{L}_1\mathcal{L}_2\Phi)(x) = \sum_{y:f_2f_{1y}=x} g_2(f_1y)g_1(y)\Phi(y). \tag{1.14}$$

Finally, by analogy with Eq.(1.12), it is possible to obtain

$$tr\mathcal{L} = \sum_{x\in Fixf} \frac{g(x)}{|\det(1 - D_xf^{-1}(x))|} \tag{1.15}$$

with D_xf being derivative of f acting in the tangent space $T_x\mathcal{M}$ and the graph of f is required to be transversal to the diagonal $\Delta \subset \mathcal{M}\times\mathcal{M}$. Eq.(1.15) coincides with that obtained in the work by Atiyah and Bott [15][4]. These authors make several additional important (for the purposes of this work) observations to be discussed now. Ruelle, Ref. [17], uses essentially the same type of arguments. These are as follows. Define the *local* Lefschetz index $\mathcal{L}_x(f)$ by

$$\mathcal{L}_x(f) = \frac{\det(1 - D_xf(x))}{|\det(1 - D_xf(x))|} \tag{1.16}$$

where $x \in Fixf$, then, as in our previous work, Ref.[2], the *global* Lefschetz index $\mathcal{L}(f)$ is defined by

$$\mathcal{L}(f) = \sum_{f(x)=x} \mathcal{L}_x(f). \tag{1.17}$$

Taking into account that $\det(1 - D_xf(x))=\prod_{i=1}^{d}(1-\alpha_i)$, where α_i are the eigenvalues of the Jacobian matrix, the determinant can be rewritten in the following useful form [19], page 133,

$$\det(1 - D_xf(x)) = \prod_{i=1}^{d}(1-\alpha_i) = \sum_{k=1}^{d}(-1)^k e_k(\alpha_1, ..., \alpha_d) \tag{1.18}$$

where the elementary symmetric polynomial $e_k(\alpha_1, ..., \alpha_d)$ is defined by

$$e_k(\alpha_1, ..., \alpha_d) = \sum_{1\leq\ i_1<\cdots<\ i_k\leq d} \alpha_{i_1}\cdots\alpha_{i_k} \tag{1.19}$$

[4]They use D_xf instead of D_xf^{-1} which makes no difference for fixed points and invertible functions. The important (for chaotic dynamics) non invertible case is discussed by Ruelle also but the results are not much different.

with $e_{k=0}=1$.With help of these results the local Lefschetz index, Eq.(1.16), can be rewritten alternatively as follows

$$\mathcal{L}_x(f)=\frac{\sum\limits_{k=1}^{d}(-1)^k e_k(\alpha_1,...,\alpha_d)}{|\det(1-D_xf(x))|}\equiv\frac{\sum\limits_{k=1}^{d}(-1)^k tr(\wedge^k D_xf(x))}{|\det(1-D_xf(x))|} \tag{1.20}$$

with $\wedge^k$denoting the k-th power of the exterior product. With help of these results, following Ruelle [17], the transfer operator $\mathcal{L}^{(k)}$ can be defined analogously to $\mathcal{L}$ in Eq.(1.15), i.e.

$$tr\mathcal{L}^{(k)}=\sum_{x\in Fixf}\frac{g(x)tr(\wedge^k D_xf(x))}{|\det(1-D_xf^{-1}(x))|} \tag{1.21}$$

and, accordingly, in view of Eq.s(1.16)-(1.21), we obtain as well

$$\sum_{k=0}^{d}(-1)^k tr\mathcal{L}^{(k)}=\sum_{x\in Fixf}g(x)\mathcal{L}_x(f). \tag{1.22}$$

Finally, if in the above formulas we replace $Fixf$ by $Fixf^n$ we have to replace $tr\mathcal{L}^{(k)}$ by $tr\mathcal{L}_n^{(k)}$. Since

$$\exp(\sum_{n=1}^{\infty}\frac{tr(\mathbf{A}^n)}{n}t^n)=[\det(\mathbf{1}-t\mathbf{A})]^{-1} \tag{1.23}$$

it is convenient to combine this result with Eq.(1.21) in order to obtain the following zeta function:

$$\begin{aligned} Z(t) &= \exp(\sum_{n=1}^{\infty}\frac{t^n}{n}\left\{\sum_{k=0}^{d}(-1)^k tr\mathcal{L}_n^{(k)}\right\}) \\ &= \prod_{k=0}^{d}\left[\exp(\sum_{n=1}^{\infty}\frac{tr\mathcal{L}_n^{(k)}}{n}t^n)\right]^{(-1)^k} \\ &= \prod_{k=0}^{d}\left[\det(\mathbf{1}-t\mathcal{L}^{(k)})\right]^{(-1)^{k+1}}. \end{aligned} \tag{1.24}$$

The final result coincides with that obtained by Ruelle as required. Thus obtained zeta function possess dynamical, number-theoretic and algebro-geometric interpretation. Such type zeta function was discussed in our earlier work, Ref.[2], in connection with dynamical/number-theoretic interpretation of the p-adic Veneziano amplitudes. The discussion above provides support in favor of earlier obtained dynamical interpretation of p-adic Veneziano amplitudes. The crucial Eq.(5.29a) of Section 5 provides additional support in favor of such interpretation.

In the meantime, based on the paper by Atiyah and Bott [15], we would like to demonstrate that Eq.(1.15) is actually the Weyl's character formula [16]. To prove that this is the case is not entirely trivial. The Appendix A provides some general background on Weyl-Coxeter reflection groups needed for understanding of the arguments presented below.

1.4 From Atiyah-Bott-Lefschetz fixed point formula to character formula by Weyl

We begin with observation that earlier obtained Eq.s (1.12) and (1.15) are essentially the same. In addition, Eq.s(1.12) and (1.7) are equivalent. Because of this, it is sufficient to demonstrate that the r.h.s of Eq.(1.7) indeed coincides with the Weyl's character formula. Although Eq.(1.7) (and, especially, Eq.s(1.15),(1.21)) looks similar to that obtained in the paper by Atiyah and Bott [15], part I, page 379, leading to the Weyl character formula (Eq.(5.12), Ref. [15], part II) neither Eq.(1.7) nor Eq.(5.11) of Atiyah and Bott paper, Ref. [15], part II, provide immediate connection with their Eq.(5.12). Hence, the task now is to restore some missing links.

To this purpose, following Bourbaki [7], consider a finite set of formal symbols $e(\mu)$ possessing the same multiplication properties as the usual exponents[5], i.e.

$$e(\mu)e(\nu) = e(\mu+\nu), \ \ [e(\mu)]^{-1} = e(-\mu) \text{ and } e(0) = 1. \tag{1.25}$$

Such defined set of formal exponents is making a free $\mathbf{Z}$ module with the basis $e(\mu)$. Subsequently we shall require that $\mu \in \Delta$ with Δ defined in the Appendix A. Suppose also that we are having a polynomial ring A[$\mathbf{X}$] made of all linear combinations of terms $\mathbf{X}^{\mathbf{n}} \equiv X_1^{n_1} \cdots X_d^{n_d}$ with $n_i \in \mathbf{Z}$, then one can construct another ring A[P] made of linear combinations of elements $e(\mathbf{p}\cdot\mathbf{n})$ with $\mathbf{p}\cdot\mathbf{n}$ $=p_1n_1 + \cdots p_dn_d$. Clearly, the above rings are isomorphic. Let $x = \sum\limits_{p\in P} x_p e(p) \in$A[P] with P=$\{p_1, ..., p_d\}$, then using Eq.(1.25) we obtain,

$$\begin{aligned}
x\cdot y &= \sum_{s\in P} x_s e(s) \sum_{r\in P} y_r e(r) = \sum_{t\in P} z_t e(t) \text{ with} \\
z_t &= \sum_{s+r=t} x_s y_r \text{ and, accordingly,} \\
x^m &= \sum_{t\in P} z_t e(t) \text{ with } z_t = \sum_{s+\cdots+r=t} x_s \cdots y_r,\ m\in\mathbf{N}
\end{aligned} \tag{1.26}$$

with $\mathbf{N}$ being some non negative integer. Introduce now the determinant of $w \in W$ via rule:

$$\det(w) \equiv \varepsilon(w) = (-1)^{l(w)}, \tag{1.27}$$

where all notations are taken from the Appendix A. If, in addition, we would require

$$w(e(p)) = e(w(p)) \tag{1.28}$$

[5] In the case of **usual** exponents it is being assumed that all the properties of formal exponents are transferable to the usual ones.

then, all elements of the ring A[P] are subdivided into two classes defined by

$$w(x) = x \text{ (invariance)} \tag{1.29a}$$

and

$$w(x) = \varepsilon(x) \cdot x \text{ (anti invariance).} \tag{1.29b}$$

These classes are very much like subdivision into bosons and fermions in quantum mechanics. All anti invariant elements can be built from the basic anti invariant element $J(x)$ defined by

$$J(x) = \sum_{w \in W} \varepsilon(w) \cdot w(x). \tag{1.30}$$

From the definition of P and from Appendix A it should be clear that the set P can be identified with the set of reflection elements w of the Weyl group W. Therefore, for all $x \in$ A[P] and $w \in W$ we obtain the following chain of equalities:

$$w(J(x)) = \sum_{v \in W} \varepsilon(v) \cdot w(v(x)) = \varepsilon(w) \sum_{v \in W} \varepsilon(v) \cdot v(x) = \varepsilon(w) J(x) \tag{1.31}$$

as required. Accordingly, *any* anti invariant element x can be written as $x = \sum\limits_{l \in P} x_p J(\exp(p))$. The denominator of Eq.(1.7), when properly interpreted with help of the results of the Appendix A, can be associated with $J(x)$. Indeed, without loss of generality let us choose the constant $\mathbf{c}$ as $\mathbf{c} = \{1, ..., 1\}$. Then, for fixed v the denominator of Eq.(1.7) can be rewritten as follows:

$$\begin{aligned}
\prod_{i=1}^{d} (1 - \exp\{-u_i^v\}) &\equiv \prod_{\alpha \in \Delta^+} (1 - \exp(-\alpha)) \equiv \tilde{d} \exp(-\rho), && (1.32) \\
\text{where } \rho &= \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha \text{ and} \\
\tilde{d} &= \prod_{\alpha \in \Delta^+} (\exp(\frac{\alpha}{2}) - \exp(-\frac{\alpha}{2})). && (1.33)
\end{aligned}$$

To prove that thus defined $\tilde{d}$ belongs to the anti invariant subset of A[P] is not difficult. Indeed, consider the action of a reflection r_i on $\tilde{d}$. Taking into account that $r_i(\alpha_i) = -\alpha_i$ we obtain

$$\begin{aligned}
r_i(\tilde{d}) &= (\exp(-\frac{\alpha_i}{2}) - \exp(\frac{\alpha_i}{2})) \prod_{\substack{\alpha \neq \alpha_i \\ \alpha \in \Delta^+}} (\exp(\frac{\alpha}{2}) - \exp(-\frac{\alpha}{2})) \\
&= -\tilde{d} \equiv \varepsilon(r_i) \tilde{d} && (1.34)
\end{aligned}$$

Hence, clearly,

$$\tilde{d} = \sum_{p \in P} x_p J(\exp(p)). \tag{1.35}$$

Moreover, it can be shown [7] that, actually, $\tilde{d} = J(\exp(\rho))$ which, in view of Eq.s(1.32), (1.33), produces identity originally obtained by Weyl :

$$\hat{d}\exp(-\rho) = \prod_{\alpha\in\Delta^+}(1-\exp(-\alpha)). \tag{1.36}$$

Applying reflection w to the above identity while taking into account Eq.s(1.28),(1.34) produces:

$$\prod_{\alpha\in\Delta^+}(1-\exp(-w(\alpha))) = \exp(-w(\rho))\ \varepsilon(w)\ \hat{d}. \tag{1.37}$$

The result just obtained is of central importance for the proof of Weyl's formula. Indeed, in view of Eq.s (1.28) and (1.37), inserting the identity : $1=\dfrac{w}{w}$ into the sum over the vertices on the r.h.s.of Eq.(1.7) and taking into account that: a) $\varepsilon(w)=\pm 1$ so that $[\varepsilon(w)]^{-1}=\varepsilon(w)$; b) actually, the sum over the vertices is the same thing as the sum over the members of the Weyl-Coxeter group (since all vertices of the integral polytope can be obtained by use of the appropriate reflections applied to the highest weight vector pointing to chosen vertex), we obtain the Weyl's character formula:

$$tr\mathcal{L}(\lambda) = \frac{1}{\hat{d}}\sum_{w\in\Delta}\varepsilon(w)\ \exp\{w(\lambda+\rho)\}. \tag{1.38}$$

It was obtained with help of the results of Appendix A, Eq.s(1.15),(1.28) and (1.37). Looking at the l.h.s. of Eq.(1.7) we can, of course, replace $tr\mathcal{L}(\lambda)$ by the sum in the l.h.s. of Eq.(1.7) if we choose the constant **c** as before. This is not too illuminating however as we shall explain now.

Indeed, since $J(x)$ in Eq.(1.30) is by construction the basic anti invariant element and the r.h.s of Eq.(1.38) is by design manifestly invariant element of A[P], it is only natural to look for the basic invariant element analogue of $J(x)$. Then, clearly, $tr\mathcal{L}(\lambda)\equiv\chi(\lambda)$ should be expressible as follows

$$\chi(\lambda) = \sum_{w\in W} n_w(\lambda)\ e(w). \tag{1.39}$$

This result admits clear physical interpretation. Indeed, the partition function Ξ of any quantum mechanical system can be represented as

$$\Xi = \sum_n g_n \exp\{-\beta E_n\} \equiv tr(\exp(-\beta\hat{H})) \tag{1.40}$$

where $\hat{H}$ is the quantum Hamiltonian of the system, β is the inverse temperature and g_n is the degeneracy factor. If we introduce the density of states $\rho(E)$ via

$$\rho(E) = \sum_n \delta(E-E_n) \tag{1.41}$$

then, the partition function Ξ can be written as the Laplace transform

$$\Xi(\beta) = \int_0^{\infty} dE \rho(E) \exp\{-\beta E\}. \tag{1.42}$$

Clearly, Eq.(1.39) is just the discrete analogue of Eq.(1.42) so that it does have a statistical/quantum mechanical interpretation as partition function. From condensed matter physics it is known that all important statistical information is contained in the density of states. Its calculation is of primary interest in physics. Evidently, the same should be true in the present case as well. Indeed, the results of Section 5 illustrate this point extensively. The density of states $n_w(\lambda)$ is known in group theory as Kostant's multiplicity formula [20]. Cartier [21] had simplified the original derivation by Kostant to the extent that it is worth displaying it here since it supplements nicely what was presented already and to be discussed further in the rest of the paper. Cartier noticed that the denominator of the Weyl character formula, Eq.(1.38), can be formally expanded with help of Eq.(1.36) as follows:

$$\left[\exp(\rho) \prod_{\alpha \in \Delta^+} (1 - \exp(-\alpha))\right]^{-1} = \sum_{w' \in W} P(w') e(-\rho - w') \tag{1.43}$$

By combining Eq.s(1.38),(1.39) and (1.43) we obtain

$$\sum_{w \in W} n_w(\lambda)\; e(w) = \sum_{w \in W} \varepsilon(w)\; \exp\{w(\lambda + \rho)\} \sum_{w' \in W} P(w') e(-\rho - w'). \tag{1.44}$$

Comparing the left side with the right we obtain finally the Kostant multiplicity formula

$$n_w(\lambda) = \sum_{w' \in W} \varepsilon(w') P(w'(\lambda + \rho) - (\rho + w)). \tag{1.45}$$

The obtained formula allows us to determine the density of states $n_w(\lambda)$ provided that we can obtain the function P explicitly. This will be discussed in Section 5.

Next, the obtained results allow us to clear up yet another problem: when compared with the r.h.s. of Eq.(5.11) of Atiyah and Bott paper, Ref.[15], part II, the r.h.s. of our Eq.(1.7) is not looking the same. We would like to explain that, actually, these expressions are equivalent. To this purpose, let us reproduce Eq.(5.11) of Atiyah and Bott first. Actually, for this purpose it is more convenient to use the paper by Bott, Ref.[16], Eq.(28). In terms of notations taken from this reference we have:

$$trace\ T_g = \sum_{\substack{w \in W \\ \alpha < 0}} \left[\frac{\lambda}{\prod (1 - \alpha)}\right]^w. \tag{1.46}$$

Comparing with the r.h.s. of our Eq.(1.7) and taking into account Eq.(1.28), the combination λ^w in the numerator of Eq.(1.46) is the same thing as $\exp\{w\lambda\}$

in Eq.(1.38). As for the denominator, Bott uses the same Eq.(1.36) as we do so that it remains to demonstrate that

$$\left[\prod_{\alpha\in\Delta^+}(1-\exp(-\alpha))\right]^w = \exp(-w(\rho))\ \varepsilon(w)\ \hat{d}. \tag{1.47}$$

In view of Eq.(1.36), we need to demonstrate that

$$\left[\hat{d}\exp(-\rho)\right]^w = \exp(-w(\rho))\ \varepsilon(w)\ \hat{d}, \tag{1.48}$$

i.e. that $\left[\hat{d}\right]^w = \varepsilon(w)\ \hat{d}$. In view of Eq.(1.34), this requires us to assume that $\left[\hat{d}\right]^w = w\hat{d}$. But, in view of Eq.s(1.28), (1.30) and (1.35), we conclude that this is indeed the case. This proves the fact that Eq.(1.46), that is Eq.(5.11) of Ref.[15], is indeed the same thing as the Weyl's character formula, Eq.(5.12) of Ref.[15], or Eq.(1.38) above. According to Kac, Ref.[12], page 174, the classical Weyl character formula, Eq.(1.38), is formally valid for both finite dimensional semisimple Lie algebras and infinite dimensional affine Kac-Moody algebras. This circumstance and the Proposition A.1 of Appendix A play important motivating role in developments presented in this work. Our treatment so far had been limited only to the $d-$ dimensional hypercube. This deficiency can be easily corrected with help of the concept of *zonotope* which we would like to explain now.

2 From zonotopes to fans and toric varieties

2.1 From zonotopes to fans

The concept of zonotope is actually not new. According to Coxeter [22] it belongs to 19th century Russian crystallographer Fedorov. Nevertheless, this concept has been truly appreciated only relatively recently in connection with oriented matroids . For our purposes it is sufficient to consider only the most elementary properties of zonotopes. To this purpose, following Ref.[23] let us consider a $p-$dimensional cube C_p defined by

$$C_p = \{\mathbf{x}\in\mathbf{R}^p, -1\leq x_i\leq 1\ ,\ i=1-p\} \tag{2.1}$$

and the surjective map $\pi:\mathbf{R}^p\rightarrow\mathbf{R}^d$ given by $\pi:\mathbf{x}\rightarrow V\mathbf{x}+\mathbf{z}$ with V being $d\times p$ matrix written in the vector form as $V=\{\mathbf{v}_1,...,\mathbf{v}_p\}$ so that, actually, $\mathbf{x}'=\mathbf{z}+\sum\limits_{i=1}^{p}x_i\mathbf{v}_i$ with $\mathbf{x}'\in\mathbf{R}^d$ and $-1\leq x_i\leq 1$. A *zonotope* $Z(V)$ is the image of a $p-$cube under affine projection π, i.e. $Z(V)$=$VC_p+\mathbf{z}$ It can be shown [23] that such created zonotope is a centrally symmetric $d-$polytope. Because of central symmetry, it is sometimes convenient to associate with such

Figure 1: Two dimensional zonotope Z from three dimensional cube C_3

d-polytope its fan. This concept can be easily understood with help of pictures. In particular, the simplest zonotope construction is displayed in Fig.1.

The fan associated with such planar zonotope is schematically displayed in Fig.2. As Fig.2 suggests, such fan is a collection of cones. These cones can be constructed by drawing semi- infinite strips whose semi- infinite edges are perpendicular to the corresponding edges of zonotope and then, by identifying the set of cones with the complement of zonotope Z (together with these strips) in $\mathbf{R}^2$. Extension of such procedure to dimensions $d \geq 2$ is straightforward. Thus formed *normal* fan [23] is *complete* if upon parallel translation of apexes of the cones to one point in $\mathbf{R}^2$(or $\mathbf{R}^d$ in general case) the resulting picture spans $\mathbf{R}^2$ (or $\mathbf{R}^d$) as depicted in Fig.2.

Figure 2: Construction of the complete fan associated with two dimensional zonotope Z

For zonotopes, in view of their central symmetry, the fans thus constructed are always complete. For the purposes of development, we would like to give more formal mathematical definitions now.

Definition.1. Let $\mathcal{R} \subset \mathbf{R}^d$ be a subset of $\mathbf{R}^d$ made of finite set of vectors

$\mathbf{y}_1, ..., \mathbf{y}_k$. Then, a *convex polyhedral* cone σ is a set

$$\sigma = \sum_{i=1}^{k} r_i \mathbf{y}_i \ \in \mathcal{R} \ , \, r_i \geq 0. \tag{2.2}$$

The scalar product, Eq.(1.6), allows us to introduce

Definition 2. The *dual* cone $\sigma^\vee$ is defined as

$$\sigma^\vee == \{\mathbf{u} \in \mathcal{R}^* : \ < \mathbf{u}, \mathbf{y} > \geq 0 \text{ for all } \mathbf{y} \in \sigma\}. \tag{2.3}$$

Finally, we have as well

Definition 3. A *face* τ of σ as an intersection of σ with any *supporting* hyperplane H_α defined by $< \mathbf{u}_\alpha, \mathbf{y} >= 0$, i.e.

$$\tau = \sigma \cap \mathbf{u}_\alpha = \{\mathbf{y} \in \sigma : \ < \mathbf{u}_\alpha, \mathbf{y} >= 0\} \tag{2.4}$$

for some $\mathbf{u}_\alpha$ in $\sigma^\vee$.

By analogy with Eq.(1.6), if vectors $\mathbf{y}_1, ..., \mathbf{y}_k \in \mathbf{Z}^d$ the cone is called *rational polyhedral*. We shall assume that this is the case from now on.

In view of Eq.(2.4), there is a correspondence between the faces and the supporting hyperplanes. Fig.2 helps us to realize that assembly of rational polyhedral cones is made out of complements of these supporting hyperplanes. It can be proven, e.g. [24], page 144, or Appendix A (part c)), that these hyperplanes intersect each other just in one point (apex) which can be chosen as origin in $\mathbf{Z}^d$. Fig.2 provides an intuitive support to this claim. Clearly, as results of Appendix A suggest, the polyhedral cones are just the chambers and the supporting hyperplanes are the reflecting hyperplanes. Fig.2 helps to visualize the interrelationship between the polytopes and the chamber system. All this, surely, can be made absolutely rigorous so that we refer our readers to literature [7,24,25]. In this work, we would like to discuss only the results of immediate relevance. In particular, the interrelationships between the affine and the projective toric varieties and the rational polyhedral cones.

We begin with a couple of definitions.

Definition 4. *A semi-group S* that is a non-empty set with associative operation is called *monoid* if it is commutative, satisfies cancellation law *(i.e.s+x=t+x* implies *s=t for all s,t,$x$$\in$ S*) and has zero element (*i.e.*$s + 0 = s, s \in S$).

Definition 5. *A monoid S is finitely generated* if exist set $a_1, ..., a_k \in S$, called *generators*, such that

$$S = Z_{\geq 0} a_1 + \cdots + Z_{\geq 0} a_k. \tag{2.5}$$

Taking into account these definitions, it is clear that the monoid $\mathbf{S}_\sigma = \sigma \cap \mathbf{Z}^d$ for the rational polyhedral cone is finitely generated.

Next, we consider a polynomial

$$f(\mathbf{z}) = f(z_1, ..., z_n) = \sum_{\mathbf{i}} \lambda_{\mathbf{i}} \mathbf{z}^{\mathbf{i}} = \sum_{\mathbf{i}} \lambda_{i_1 ..., i_n} z_1^{i_1} \cdots z_n^{i_n}, \ (\lambda_{\mathbf{i}}, \ z_m^{i_m} \in \mathbf{C}, 1 \leq m \leq n). \tag{2.6}$$

It belongs to the polynomial ring K[$\mathbf{z}$] closed under ordinary addition and multiplication.

Definition 6. An *affine* algebraic variety $V \in \mathbf{C}^n$ is the set of zeros of collection of polynomials from the above ring.

Collection of such polynomials is **finite** according to famous Hilbert's Nullstellensatz and forms the set $I(\mathbf{z}) := \{f \in \mathrm{K}[\mathbf{z}],\ f(\mathbf{z}) = 0\}$ of maximal ideals usually denoted *spec*K[$\mathbf{z}$].

Definition 7. Zero set of a *single* function belonging to $I(\mathbf{z})$ is called *algebraic hypersurface* so that the set $I(\mathbf{z})$ corresponds to the *intersection* of a finite number of hypersurfaces.

2.2 From fans to affine toric varieties

To connect these results with those introduced earlier, let us consider now the set of Laurent monomials of the type $\lambda z^{\boldsymbol{\alpha}} \equiv \lambda z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. We shall be particularly interested in *monic* monomials for which $\lambda = 1$ because such monomials form a closed polynomial subring with respect to usual multiplication and addition. The crucial step forward is to assume that the exponent $\boldsymbol{\alpha} \in \mathbf{S}_\sigma$. This allows us to define the following mapping

$$u_i := z^{a_i} \tag{2.7}$$

with a_i being one of generators of the monoid $\mathbf{S}_\sigma$ and $z \in \mathbf{C}$. Such mapping provides an isomorphism between the additive group of exponents a_i and the multiplicative group of monic Laurent polynomials. Let us recall now that the function ϕ is considered to be *quasi homogenous* of degree d with exponents $l_1, ..., l_n$ if

$$\phi(\lambda^{l_1} x_1, ..., \lambda^{l_n} x_n) = \lambda^d \phi(x_1, ..., x_n), \tag{2.8}$$

provided that $\lambda \in \mathbf{C}^*$. Applying this result to $z^{\mathbf{a}} \equiv z_1^{a_1} \cdots z_n^{a_n}$ we reobtain equation for the cone

$$\sum_j (l_j)_i \, a_j = d_i \tag{2.9}$$

which belongs to the monoid $\mathbf{S}_\sigma$. Here the index i is numbering different monomials. Clearly, the same result can be achieved if instead we would consider products of the type $u_1^{l_1} \cdots u_n^{l_n}$ and rescale all $z_i' s$ by the same factor λ. Eq(2.9) is actually a scalar product with $(l_j)_i$ living in the space *dual* to $a_j' s$. So that, in view of Eq.(2.3), the set of $(l_j)_i' s$ can be considered as the set of generators for the dual cone $\sigma^\vee$. Next, in view of Eq.(2.6) let us consider polynomials of the type

$$f(\mathbf{z}) = \sum_{\mathbf{a} \in \mathbf{S}_\sigma} \lambda_{\mathbf{a}} \mathbf{z}^{\mathbf{a}} = \sum_{\mathbf{l}} \lambda_{\mathbf{l}} \mathbf{u}^{\mathbf{l}}. \tag{2.10}$$

They form a polynomial ring as before. The ideal for this ring can be constructed based on the observation that for the fixed d_i and the assigned set of cone

generators a_i' there is more than one set of generators for the dual cone. This redundancy produces relations of the type

$$u_1^{l_1} \cdots u_k^{l_k} = u_1^{\tilde{l}_1} \cdots u_k^{\tilde{l}_k}. \tag{2.11}$$

If now we require $u_i \in \mathbf{C}_i$, then it is clear that the above equation belongs to the ideal $I(\mathbf{z})$ of the above polynomial ring and that Eq.(2.11) represents the hypersurface. As before, $I(\mathbf{z})$ represents the intersection of these hypersurfaces thus forming the affine *toric* variety $X_{\sigma^\vee}$. The generators $\{u_1, ..., u_k\} \in \mathbf{C}^k$ are coordinates for $X_{\sigma^\vee}$. They represent the same point in $X_{\sigma^\vee}$ if and only if $\mathbf{u}^{\mathbf{l}} = \mathbf{u}^{\tilde{\mathbf{l}}}$. Thus formed toric variety corresponds to just one (dual) cone.

2.3 Building toric varieties from affine toric varieties

As Appendix A suggests, there is one- to- one correspondence between cones and chambers. From chambers one can construct a gallery and, hence, a building. Accordingly, information leading to the design of particular building can be used for construction of toric variety from the set of affine toric varieties. To do so one only needs the set of *gluing maps* $\{\ \Psi_{\sigma^\vee \breve{\sigma}^\vee}\}$. Thus, we obtain the following

Definition 8. Let Σ be the complete fan and $\coprod_{\sigma^\vee \in \Sigma} X_{\sigma^\vee}$ be the disjoint union of affine toric varieties. Then, using the set of gluing maps $\{\ \Psi_{\sigma^\vee \breve{\sigma}^\vee}\}$ such that each of them identifies two points $x \in X_{\sigma^\vee}$ and $\breve{x} \in X_{\breve{\sigma}^\vee}$ on respective affine varieties, one obtains the toric variety X_Σ determined by fan Σ.

Remark 1.Thus constructed variety X_Σ may contain singularities which should be obvious just by looking at Eq.(2.11). There is a procedure of desingularization described, for example in Ref.[26]. In this work (for the sake of space) we are going to bypass this circumstance. This is permissible because in the end the results are going to come up the same as for nonsingular varieties.

2.4 Torus action and its invariants

Based on the results just presented we can accomplish more. We begin with

Definition 9. The set $T := (\mathbf{C}\backslash 0)^n =: (\mathbf{C}^*)^n$ is called complex algebraic torus.

Since each $z \in \mathbf{C}^*$ can be written as $z = r\exp(i\theta)$ so that for each $r > 0$ the fiber: $\{z \in \mathbf{C}^* \mid |z| = r\}$ is a circle of radius r, we can represent T as the product $(R_{>0})^n \times \left(S^1\right)^n$. The product of n circles $\left(S^1\right)^n$ which is the deformation retract of T is indeed a topological torus. Following Fulton [26], we are going to call it a *compact torus* S_n. So that the algebraic torus is a product of a compact torus and a vector space. This circumstance is helpful since whatever we can prove for the deformation retract can be extended to the whole torus T. This explains the name "algebraic torus". Let now G be the group acting (multiplicatively) on the set X via mapping $G \times X \to X$, i.e. $(g, x) \to gx$ provided that for all $g, h \in G$, $g(hx) = (gh)x$ and $ex = x$ for some unit element e of G. The subset $Gx := \{gx \mid g \in X\}$ of X is called the *orbit* of x. Denote by H the subgroup

of G that fixes x, i.e. $H := \{gx = x \mid g \in X\}$ the *isotropy* group. Surely, there could be more than one fixed point for equation $gx = x$ and all of them are conjugate to each other. A *homogenous* space for G is the subspace of X on which G acts without fixed points. The crucial step forward can be made by introducing a concept of an *algebraic* group [27].

Definition 10. A *linear algebraic group G* is a) an affine algebraic variety and b) a group in the sense given above, i.e.

$$\mu \quad : \quad G \times G \to G \ ; \ \mu(x,y) = xy \tag{2.12a}$$

$$i \quad : \quad G \to G \ ; \iota(x) = x^{-1}. \tag{2.12b}$$

Remark 2. It can be shown [28], page150, that G as linear algebraic group is isomorphic (as an algebraic group) to a closed subgroup of $GL_n(\mathrm{K})$ for some $n \geq 1$ and any closed field K such as $\mathbf{C}$ or $p-$adic.

This fact plays a crucial role in the whole development given below. Moreover, another no less important direction of development comes from

Remark 3. Let G be a linear algebraic group, $A = \mathrm{K}[G]$ its affine algebra, then, the set of rules given by Eq.s(2.12) can be replaced by the following set

$$\mu^* \quad : \quad A \to A \otimes_K A \ \text{ comultiplication} \tag{2.13a}$$

$$\iota^* \quad : \quad A \to A \text{ taking the antipode of } A. \tag{2.13b}$$

Thus, study of properties of linear algebraic groups is equivalent to study of coassociative Hopf algebras leading to quantum groups [27,28], etc.

In this work we are not going to develop this line of thought however, except for few comments made in Section 6. Rather, we would like to connect previous discussion with Remark 2. To this purpose, in accord with earlier results, we need to introduce the following

Definition 11. The *torus action* is a continuous map : $T \times X_\Sigma \to X_\Sigma$ such that for each affine variety corresponding to the dual cone it is given by

$$T \times X_{\sigma^\vee} \to X_{\sigma^\vee} \ , \ (t,x) \mapsto tx := (t^{a_1}x_1, ..., t^{a_k}x_k). \tag{2.14}$$

Naturally, such action should be compatible with gluing maps thus extending it from one cone (chamber) to the entire variety X_Σ(building). The compatibility is easy to enforce since for each of Eq.s (2.11) multiplication by $t-$factors will not affect the solutions set. This can be formally stated as follows. Let Ψ : $X_\Sigma \to X_{\tilde{\Sigma}}$ be a map and $\alpha : T \to T'$ a homomorphism, then the map Ψ is called *equivariant* if it obeys the following identity

$$\Psi(cx) = \alpha(c)\Psi(x) \text{ for all } c \in T. \tag{2.15}$$

Naturally, $\alpha(c)$ is character of the algebraic torus group.[6] For physical applications it is more advantageous, following Stanley [31], to consider *invariants* given by

$$\Psi(cx) = \Psi(x) \tag{2.16}$$

[6]This fact is known as Borel-Weil theorem [29]. As such it belongs to the theory of induced group representations [30].

rather than $\alpha-$invariants (equivariants) in Stanley's terminology. To obtain invariants we need to study the orbits of the torus action first. To this purpose, in view of Eq.(2.14), we need to consider the following fixed point equation

$$t^{a}x=x. \tag{2.17}$$

Apart from trivial solutions: $x=0$ and $x=\infty$, there is a nontrivial solution $t^{a}=1$ for any x. For integer $a's$ this is a cyclotomic equation whose nontrivial $a-1$ solutions all lie on the circle S^{1}. In view of this circumstance, it is possible to construct invariants for this case as we are going to explain now. First, such an invariant can be built as a *ratio* of two equivariant mappings of the type given by Eq.(2.15). By construction, such ratio is the *projective* toric variety. Unlike the affine case, such varieties are **not** represented by functions of homogenous coordinates in $\mathbf{CP}^{n}$. Instead, they are just constants associated with points in $\mathbf{CP}^{n}$ which they represent. Second option, is to restrict the algebraic torus to compact torus acting on the circle. These two options are interrelated in important way as we would like to explain now.

To this purpose we notice that Eq.(2.14) still holds if some of $t-$factors are replaced by 1's. This means that one should take into account all situations when one, two, etc. t-factors in Eq.(2.14) are replaced by 1's and account all permutations involving such cases. This leads to torus actions on toric subvarieties. It is very important that different orbits belong to different subvarieties which do not overlap. Thus, by design, X_{Σ} is the disjoint union of finite number of orbits which are identified with subvarieties of X_{Σ}. To make all this more interesting, let us consider instead of Eq.(2.14) its transpose. Then, it can be viewed as part of the eigenvalue problem

$$\begin{pmatrix} t^{a_1} & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & t^{a_k} \end{pmatrix}\begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_k \end{pmatrix}=\begin{pmatrix} t^{a_1}x_1 \\ \cdot \\ \cdot \\ \cdot \\ t^{a_k}x_k \end{pmatrix}. \tag{2.18}$$

Under such conditions the vector $(x_1,...,x_k)$ forms a basis of $k-$dimensional vector space V so that the vector $(x_1,...,x_i)$, $i\leq k$, forms a basis of subspace V_i . This allows us to introduce a complete flag f_0 of subspaces in V (just like in our previous work, Ref.[32])

$$f_0:0=V_0\subset V_1\subset...\subset V_k=V. \tag{2.19}$$

Consider now action of G on f_0 . Taking into account the Remark 2, effectively, G=$GL_n(K)$. The matrix representation of this group possess remarkable property. To formulate this property requires several definitions. They are provided below. To make our exposition less formal, we would like to explain their rationale now so that their importance cannot be underestimated.

The eigenvalue problem for some matrix $\mathbf{A}$ is reduced to finding eigenvectors and eigenvalues of the equation $\mathbf{Ax}$=$\boldsymbol{\lambda}\mathbf{x}$. Eq.(2.18) fits trivially to this category

of problems. Suppose now that the matrix $\mathbf{A}$ can be presented as $\mathbf{A}$=UTU^{-1} with U being some matrix which we need to find, then, of course, the above eigenvalue equation can be written as TU$^{-1}\mathbf{x}$ =U$^{-1}\mathbf{x}$ and, as usual, if we replace U$^{-1}\mathbf{x}$ by $\mathbf{y}$, we reobtain back Eq.(2.18). This standard exercise can be made more entertaining by making an assumption (to be justified below) that equation U$^{-1}\mathbf{x} = \mathbf{x}$ or, more generally, equation U$^{-1}\mathbf{x} = \boldsymbol{\pi}\mathbf{x}$ can be solved, where $\boldsymbol{\pi}$ is some permutation of indices of $\mathbf{x}$. If this is possible, then, instead of the torus action defined by Eq.(2.14), one can consider much more general action $G \times X_{\sigma^\vee}$. Moreover, by making a quotient $G \times X_{\sigma^\vee}/H$ with H made of U$^{-1}\mathbf{x} = \mathbf{x}$ we shall obtain transitive action of G on $X_{\sigma^\vee}$ which permutes the "roots", i.e. the set $\{t^{a_1}, ..., t^{a_n}\}$. That is it acts as the Weyl reflection group and, indeed, it is the Weyl group (to be defined and explained below) for this case. In the language of linear algebraic groups, taking the quotient of the action of such group on $X_{\sigma^\vee}$ by H causes G to act transitively not on affine but on projective (quasi-projective to be exact [28], page 160) toric variety. From the definition of such varieties (e.g. read discussion next to Eq.(2.17)) it follows that such transitive action is associated with "motion" on the fixed point set of such projective variety. This fact has important physical consequences affecting the rest of this paper, especially, Section 5. In the meantime, let us return to the promised definitions. We have

Definition 12. Given that the set $GL_n(K) = \{x \in M_n(K) |\ \det x \neq 0\}$ with $M_n(K)$ being $n \times n$ matrix with entries $x_{i,j} \in K$, forms the general linear group, the matrix $x \in M_n$(K) is

a) *semisimple* ($x = x_s$), if it is diagonalizable, that is $\exists g \in GL_n(K)$ such that gxg^{-1} is a diagonal matrix;

b) *nilpotent* ($x = x_n$) if $x^m = 0$,that is for some positive integer m all eigenvalues of matrix x^m are zero;

c) *unipotent* ($x = x_u$), if $x - 1_n$ is nilpotent, i.e. x is the matrix whose only eigenvalues are 1's.

Just like with the odd and even numbers the above matrices, *if they exist,* form closed disjoint subsets of $GL_n(K)$, e.g. all $x, y \in M_n(K)$ commute; if x, y are semisimple so is their sum and product, etc. Most important for us is the following fact:

Proposition 1.*Let $x \in GL_n(K)$, then $\exists$ x_u and x_s such that $x = x_s x_u = x_u x_s$ Both $x_s$and x_u are determined by the above conditions uniquely.*

The proof can be found in Ref.[33], page96.

This proposition is in fact a corollary of the Lie-Kolchin theorem which is of central importance for us. To formulate this theorem we need to introduce yet another couple of definitions. In particular, if A and B are closed (finite) subgroups of an algebraic group G one can construct the group (A, B) made of commutators $xyx^{-1}y^{-1}$, $x \in A$, $y \in B$. With help of such commutators the following definition can be made

Definition 13. The group G is *solvable* if its derived series terminates in the unit element e. The derived series is being defined inductively by $\mathcal{D}^{(0)}G = G, \mathcal{D}^{(i+1)}G = (\mathcal{D}^{(i)}G, \mathcal{D}^{(i)}G), i \geq 0$.

Such definition implies that an algebraic group G is solvable if and only if there is exist a chain $G = G^{(0)} \supset G^{(1)} \supset \cdots G^{(n)} = e$ for which $(G^{(i)}, G^{(i)}) \subset G^{i+1}$ $(0 \leq i \leq n)$, Ref.[33], page 111. Finally,

Definition 14. The group is called *nilpotent* if $\mathcal{E}^{(n)}G = e$ for some n, where $\mathcal{E}^{(0)} = G, \mathcal{E}^{(i+1)} = (G, \mathcal{E}^{(i)}G)$.

Such group is represented by the nilpotent matrices. Based on this it is possible to prove [112] that Every nilpotent group is solvable [33],page 112.These results lead to the Lie-Kolchin theorem of major importance

Theorem 1. *Let G be connected solvable algebraic group acting on a projective variety X. Then G has a fixed point in X.*

(for proofs e.g .see Ref.[33], page113)

In view of the Remark 2, we know that such G is a subgroup of $GL_n(K)$. Moreover, $GL_n(K)$ has at least another subgroup, called *semisimple*, for which Theorem 1 does not hold.

Definition 15. The group G is *semisimple* if it has no closed connected commutative normal subgroups other than e.

Such group is represented by semisimple, i.e. diagonal (or torus) matrices while the members of the unipotent group are represented by the upper triangular matrices with all diagonal entries being equal to 1. In view of the Theorem 1, the unipotent group is also solvable and, accordingly, there must be an element B of such group which fixes the flag f_0 given by Eq.(2.19), i.e. $Bf_0 = f_0$. Let now $g \in GL_n(K)$. Then, naturally, $gf_0 = f$ where $f \neq f_0$. From here we obtain, $f_0 = g^{-1}f$. Next, we get as well $Bg^{-1}f = g^{-1}f$ and, finally, $gBg^{-1}f = f$. From here, it follows that $gBg^{-1} = \tilde{B}$ is also an element of $GL_n(K)$ which fixes flag f, etc. This means that all such elements are conjugate to each other and form the *Borel subgroup.*We shall denote all elements of this sort by B[7]. Clearly, the quotient group G/B will act transitively on X. Since this quotient is an algebraic linear group, it is also a projective variety called *flag variety*[8], Ref.[28], page 176. It should be clear by now that the group G is made out of at least two subgroups: B just described, and N. The *maximal torus* T subgroup can be defined now as $T = B \cap N$. This allows to define the Weyl group $W = N/T$. Although this group has the same name as that discussed in the Appendix A, its true meaning in the present case requires some explanations to be provided below. This is done in several steps.

First, following Appendix A, we notice that the "true" Weyl group is made of reflections, i.e. involutions of order 2. Following Tits [7], we introduce a quadruple (G, B, N, S) (the Tits system) where S is subgroup of W made of

[7]These are made of upper triangular matrices belonging to $GL_n(K)$. Surely, such matrices satisfy Proposition 1.

[8]Flag variety is directly connected with *Schubert variety* [34],page 124. Schubert varieties were considered in our work, Ref. [32], in connection with exact combinatorial solution of the Kontsevich-Witten model. This observation naturally leads to combinatorial treatment of the whole circle of problems in this work

elements such that $S = S^{-1}$ and $1 \notin S$[9]. Then, it can be shown that $G = BWB$ (*Bruhat decomposition*) and, moreover, that the Tits system is isomorphic to the Coxeter system, i.e. to the Coxeter reflection group of Appendix A. The full proof can be found in Bourbaki [7], Chr.6, paragraph 2.4.

Second, since $W = N/T$ it is of interest to see the connection (if any) between W and the quotient $G/B = BWB/B = [B\,(N/T)\,B]\,/B$. Suppose that N commutes with B, then evidently we would have $G/B \simeq (N/T)\,B$ and since B fixes the flag f we are left with action of N on the flag. Looking at Eq.(2.18) and noticing that the diagonal matrix T (the centralizer) can be chosen as a reference (identity) transformation, so that the commuting matrix N (the normalizer) should permute t^{a_i} 's . To prove that this is indeed the case requires few additional steps. To begin, using Eq.s(2.7)-(2.9),(2.11),(2.14) and (2.15) consider the map Ψ for monomial $\mathbf{u}^{\mathbf{l}} = u_1^{l_1} \cdots u_n^{l_n} \equiv z_1^{l_1 a_1} \cdots z_n^{l_{n1} a_n}$. For such map the character $c(t)$ is given by

$$c(t) = t^{<\mathbf{l}\cdot\mathbf{a}>} \tag{2.20}$$

where $< \mathbf{l} \cdot \mathbf{a} >= \sum_i l_i a_i$ and both l_i and a_i are being integers. Following Ref.[35], consider limit $t \to 0$ in the above expression. Clearly, we obtain:

$$c(t) = \begin{cases} 1 \text{ if } < \mathbf{l} \cdot \mathbf{a} >= 0 \\ 0 \text{ if } < \mathbf{l} \cdot \mathbf{a} > \neq 0 \end{cases} . \tag{2.21}$$

Evidently, the equation $< \mathbf{l} \cdot \mathbf{a} >= 0$ describes a hyperplane or, better, a set of hyperplanes for given vector $\mathbf{a}$ (Appendix A). Based on previous discussion such set forms at least one chamber. To be more accurate, following the same reference, we would like to complicate matters by introducing the subset $I \subset \{1, ..., n\}$ such that say only those $l_i' s$ which belong to this subset satisfy $< \mathbf{l} \cdot \mathbf{a} >= 0$ then, naturally, one obtains one- to- one correspondence between such subsets and earlier defined flags. Clearly, the set of thus constructed monomials forms invariant of torus group action. In Section 1 we had discussed invariance with respect to the Coxeter-Weyl reflection group, e.g. see Eq.(1.29a). It is of interest to discuss now if such set of monomials is also invariant with respect to action of the reflection group. More on this will be discussed in Section 5**.**This can be achieved if we demonstrate that the Weyl group $W = N/T$ permutes a_i's thus transitively "visiting" different hyperplanes. This will be demonstrated momentarily. Before doing this, we would like to change the rules of the game slightly. To this purpose, we shall replace the limiting $t \to 0$ procedure by requiring $t = \xi$ (e.g. see Eq.(2.17) and discussion following this equation) where ξ is nontrivial n-th root of unity. After such replacement we formally entering the domain of pseudo-reflection groups (Appendix A). The results which follow can be obtained with help of both real and pseudo reflection groups as it will become clear upon reading. Replacing t by ξ causes us to change the rule, Eq.(2.21), as follows

$$c(\xi) = \begin{cases} 1 \text{ if } < \mathbf{l} \cdot \mathbf{a} >= 0 \mod n \\ 0 \text{ if } < \mathbf{l} \cdot \mathbf{a} > \neq 0 \end{cases} \tag{2.22}$$

[9]Such soubroup always exist for compact Lie groups considered as symmetric spaces .

For reasons which will become clear in Section 4.4.2. we shall call equation $<\mathbf{l}\cdot\mathbf{a}>=0$ (or, more relaxed, $<\mathbf{l}\cdot\mathbf{a}>=n$) the *Veneziano condition* while $<\mathbf{l}\cdot\mathbf{a}>=0 \bmod n$ the *Kac condition*[10]. The results of the Appendix A (part c)) indicate that the first option is characteristic for the standard Weyl-Coxeter reflection groups while the second is characteristic for the affine Weyl-Coxeter groups thus leading to Kac-Moody affine algebras.

At this moment we are ready to demonstrate that $W = N/T$ is indeed the Weyl reflection group.[11] Although we had mentioned earlier that such proof can be found in Bourbaki [7], still, it is instructive to provide a qualitative arguments exhibiting the essence of the problem. These arguments also will be of some help for the next section. Let us begin with assembly of $(d+1)\times(d+1)$ matrices with complex coefficients. They belong to the group $GL_{d+1}(\mathbf{C})$. Consider a subset of all diagonal matrices and, having in mind the results to be presented in Sections.4 and 5, let us assume that the diagonal entries are made of $n-th$ roots of unity ξ. Taking into account the results of Appendix A (part d)) on pseudo-reflection groups, each diagonal entry can be represented by ξ^k with $1 \leq k \leq n-1$ so that there are $(n-1)^{d+1}$ different diagonal matrices- all commuting with each other. Among these commuting matrices we would like to single out those which have all $\xi^k\,'s$ the same. Evidently, there are $n-1$ of them. They are effectively the unit matrices and they are forming the centralizer of W. The rest belongs to normalizer.[12] The number $(n-1)^{d+1}/(n-1) = (n-1)^d$ was obtained earlier, e.g. see Eq.(1.5) (with n being replaced by $2n$). This is not just a mere coincidence. In Section 5 we shall provide some refinements of this result motivated by relevant physics. It should be clear already that we are discussing only the simplest possibility for the sake of illustration of general principles.

Next, let us consider just one of the diagonal matrices $\tilde{T}$ whose entries are all different and made of powers of ξ. Let $g \in GL_{d+1}(\mathbf{C})$ and consider the automorphism: $\mathcal{F}(\tilde{T}) := g\tilde{T}g^{-1}$. Along with it we would like to consider an orbit $O(\tilde{T}) := g\tilde{T}C$ where C is any of diagonal matrices which belong to earlier discussed centralizer.[13] Clearly, $O(\tilde{T}) = g\tilde{T}g^{-1}gC = \mathcal{F}(\tilde{T})gC = \mathcal{F}(\tilde{T})C$.

[10] Actually, this condition should be called Kac–Moody-Bloch-Bragg (K-M-B-B). The name Bloch comes from the Bloch equation/condition for the Schrödinger wave function of a single electron in perfect crystals [36]. It is essentially of the same nature as Eq.s (2.15),(2.16). The name Bragg comes from the Bragg condition in X-ray crystallography. The analogy with results from condensed matter physics should not be totally unexpected in view of results of Appendix A (part c)) and Eq.(1.42).We shall encounter it later in Sections 4.4.2. and 4.4.3.

[11] Since $G/B \simeq N/T$ and since G/B is the projective flag variety (footnote 8). The same should be true for N/T .This is indeed the case as was demonstrated in Ref. [37] and, using different metods, by Danilov [2] and Ewald [24]. Homology and cohomology calculations for such varieties are rather sophisticated [2,24]. Fortunately, the major results (needed for physical applications) can be obtained much simpler. This is explained below and in Section 5.

[12] As with Eq.(2.20) one can complicate matters by considering matrices which have several diagonal entries which are the same. Then, as before, one should consider the flag system where in each subsystem the entries are all different. The arguments applied to such subsystems will proceed the same way as in the main text.

[13] The presence of C factor undescores the fact that we are considering the orbit of the factorgroup $W = N/T$.

Denote now $\tilde{T}$ $=\tilde{T}$ $_1$ and consider another matrix $\tilde{T}$ $_2$ which belongs to the same set and suppose that there is such matrix g_{12} that $\tilde{T}$ $_2C = \mathcal{F}(\tilde{T})C$. If such matrix exist, it should belong to the normalizer and, naturally, the same arguments can be used to $\tilde{T}_3$, etc. Hence, the following conclusions can be drawn. If we had started with some element $\tilde{T}_1$ of maximal torus, the orbit of this element will return back and intersect the maximal torus in *finite* number of points (in our case the number of points is exactly $(n-1)^d$). By analogy with the theory of dynamical systems we can consider these intersection points of the orbit $O(\tilde{T})$ with the T-plane as Poincare$'$ crossections. Hence, as it is done in dynamical systems, we have to study the transition map between these crossections. The orbit associated with such map is precisely the orbit of the Weyl group W. It acts on these points transitively [33],page 147. Provided that such set of fixed point exists, such arguments justify dynamical interpretation of Weyl's character formula presented in Section 1. The fact that such fixed point set exist is guaranteed by the Theorem 10.6 by Borel [27]. Its proof relies heavily on the Lie-Kolchin theorem (our Theorem 1).

Mathematical results presented thus far can be made physically relevant if we close the circle of ideas by considering transition back to zonotopes. This is discussed in the next section.

3 From toric varieties back to zonotopes

3.1 Coadjoint orbits

So far we were working with Lie groups. To move forward we need to use the Lie algebras associated with these groups. In what follows, we expect familiarity with basic relevant facts about Lie groups which can be found in the books by Serre [38], Humphreys [11] and Kac [12]. First, we notice that the Lie group matrices h_i associated with the Lie group maximal tori T_i (that is with all diagonal matrices considered earlier) are commuting with each other thus forming the Cartan subalgebra, i.e.

$$[h_i, h_j] = 0. \tag{3.1}$$

The matrices belonging to the normalizer are made of two types x_i and y_i corresponding to the root systems Δ^+ and Δ^- . The fixed point analysis described at the end of previous section is translated into the following set of commutators

$$\begin{aligned}
[x_i, y_j] &= \begin{cases} h_i \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} && \text{(3.2a)} \\
[h_i, x_j] &= < \alpha_i^\vee, \alpha_j > x_j && \text{(3.2b)} \\
[h_i, y_j] &= - < \alpha_i^\vee, \alpha_j > y_j && \text{(3.2c)}
\end{aligned}$$

$i = 1, ..., n$. To insure that the matrices(operators) $x_i's$ and $y_i's$ are nilpotent (that is their Lie group ancestors belong to the Borel subgroup) one must impose

two additional constraints. According to Serre [38] these are:

$$(\mathrm{ad}x_i)^{-<\alpha_i^\vee,\alpha_j>+1}(x_j) = 0,\ i \neq j \tag{3.2d}$$

$$(\mathrm{ad}y_i)^{-<\alpha_i^\vee,\alpha_j>+1}(y_j) = 0, i \neq j. \tag{3.2e}$$

where $\mathrm{ad}_X\, Y = [X,Y]$. From the book by Kac [12] one finds that *exactly the same* relations characterize the Kac-Moody affine Lie algebra. This fact is very much in accord with general results presented in the Appendix A. It is important for our purposes to realize that for each i Eq.s(3.2a-c) can be brought to form (upon rescaling) which coincides with the Lie algebra $sl_2(\mathbf{C})$[14] and, if we replace $\mathbf{C}$ with any closed number field $\mathbf{F}$, then all semisimple Lie algebras are made of copies of $sl_2(\mathbf{F})$ [11], page 25.The Lie algebra $sl_2(\mathbf{C})$ is isomorphic to the algebra of operators acting on differential forms living on Hodge-type complex manifolds [10]. This observation is essential for physical applications presented in Section 5.4. The connection with Hodge theory can be also established through the method of coadjoint orbits which we would like to discuss now. Surely, there are other reasons to discuss this method as it will become clear upon reading the rest of this section.

We begin by considering the orbit in Lie group. It is given by the Ad operator, i.e. $O(X) = \mathrm{Ad}_g X = gXg^{-1}$ where $g \in G$ and $X \in \mathsf{g}$ with G being the Lie group and g its Lie algebra. For compact groups globally and for noncompact locally every group element g can be represented through exponential, e.g. $g(t)=\exp(tX_g)$ with $X_g \in \mathsf{g}$. Accordingly, for the orbit we can write $O(X) \equiv X(t) = \exp(tX_g)X\exp(-tX_g)$. Since the Lie group is a manifold $\mathcal{M}$, the Lie algebra forms the tangent bundle of vector fields at given point of $\mathcal{M}$. In particular, the tangent vector to the orbit $X(t)$ is determined, as usual, by $TO(X) = \frac{d}{dt}X(t)_{t=0} = [X_g, X] = ad_{X_g}X$. Now we have to take into account that, actually, our orbit is made for vector X which comes from the torus, i.e. $T = \exp(tX)$. This means that when we consider the commutator $[X_g, X]$ it will be zero for $X_{g_i} = h_i$ and nonzero otherwise. Consider now the Killing form $\kappa(x,y)$ for two elements x and y of the Lie algebra:

$$\kappa(x,y) = tr(adx\ ady). \tag{3.3}$$

From this definition it follows that

$$\kappa([x,y],z) = \kappa(x,[y,z]). \tag{3.4}$$

The roots of the Weyl group can be rewritten in terms of the Killing form [11]. It effectively serves as scalar multiplication between vectors belonging to the Lie algebra and, as such, allows to determine the notion of orthogonality between these vectors. In particular if we choose $x \rightarrow X$ and $y,z \in h_i$,then, it is clear that the vector tangential to the orbit $O(X)$ is going to be orthogonal to the subspace spanned by the Cartan subalgebra. This result can be reinterpreted

[14] This fact is known as Jacobson-Morozov theorem [9]

from the point of view of symplectic geometry due to work of Kirillov [30].To this purpose we would like to rewrite Eq.(3.4) in the equivalent form, i.e.

$$\kappa(x,[y,z]) = \kappa(x,\mathrm{ad}_y z) = \kappa(\mathrm{ad}_x^* y, z) \tag{3.5}$$

where in the case of compact Lie group $\mathrm{ad}_x^* y$ actually coincides with $\mathrm{ad}_x y$. The reason for introducing the asterisk * lies in the following chain of arguments. Already in Eqs(A.1) and (2.9) we had introduced vectors from the dual space. Such construction is possible as soon as the scalar multiplication is defined. Hence, for the orbit $\mathrm{Ad}_g X$ there must be a vector ξ in the dual space g^* such that equation

$$\kappa(\xi, \mathrm{Ad}_g X) = \kappa(\mathrm{Ad}_g^* \xi, X) \tag{3.6}$$

defines the *coadjoint orbit* $O^*(\xi)$=$\mathrm{Ad}_g^* \xi$. Accordingly, for such an orbit there is also the tangent vector $TO^*(\xi)$=$ad_{\mathsf{g}}^* \xi$ to the orbit and, clearly, we have $\kappa(\xi, ad_{X_g} X) = \kappa(ad_{\mathsf{g}}^* \xi, X)$. In the case if we are dealing with the flag space, the family of coadjoint orbits also will represent the flag space structure. Next, let $x \in \mathsf{g}^*$ and $\xi_1, \xi_2 \in TO^*(x)$, then consider the properties of the (symplectic) form $\omega_x(\xi_1, \xi_2)$ to be determined explicitly momentarily. For this one needs to introduce notations, e.g. $ad_{\mathsf{g}}^* x = f(x, \mathsf{g})$ so that for g_1 and $\mathsf{g}_2 \in \mathsf{g}$ one has $\xi_i = f(x, \mathsf{g}_i), i = 1, 2$. Then, one can claim that for compact Lie group and the associated with it Lie algebra $\omega_x(\xi_1, \xi_2) = \kappa(x, [\mathsf{g}_1, \mathsf{g}_2])$. Indeed, using Eq.(3.5) one obtains: $\kappa(x, [\mathsf{g}_1, \mathsf{g}_2]) = \kappa(\xi_1, \mathsf{g}_2) = -\kappa(x, [\mathsf{g}_2, \mathsf{g}_1]) = -\kappa(\xi_2, \mathsf{g}_1)$. Thus constructed form defines the symplectic structure on the coadjoint orbit $O^*(x)$ since it is closed, skew -symmetric, nondegenerate and is effectively independent of the choice of g_1 and g_2.The proofs can be found in the literature [39].Thus obtained symplectic manifold $\mathcal{M}_x$ is the quotient g/g_h with g_h being made of vectors of Cartan subalgebra. Clearly, for such vectors, by construction, $\omega_x(\xi_1, \xi_2) = 0$. From the point of view of symplectic geometry, such points correspond to critical points for the velocity vector field on the manifold $\mathcal{M}_x$ at which the velocity vanishes. They are in one-to one correspondence with the fixed points of the orbit $O(X)$. This fact allows us to use the Poincare-Hopf index theorem (earlier used in our works on dynamics of 2+1 gravity [40,41]) in order to obtain the Euler characteristic χ for such manifold as sum of indices of vector fields which can exist on $\mathcal{M}_x$. From the discussion presented in Section 2.4. it should be clear that χ is proportional to $(n-1)^d$. It can be proven, Ref. [42], page 259, that it is equal exactly to this number. From Section 1 it should be clear, however, that different Coxeter-Weyl reflection groups may have different numbers of this type. And, indeed, in Section 5 we shall discuss related important example of this sort.

To complete the above discussion, following work by Atiyah [43], we notice that every nonsingular algebraic variety in projective space is symplectic. The symplectic (Kähler) structure is being inherited from that of projective space. The complex Kähler structure for symplectic (Kirillov) manifold is actually of Hodge type. This comes from the following arguments. First, since we have used the Killing form to determine Kirillov's ω_x symplectic form and since the same Killing form is effectively used in Weyl reflection groups [39], e.g. see Eq.(A.1),

the induced unitary one dimensional representation of the torus subgroup of $\mathrm{GL}_n(\mathbf{C})$ is obtained according to Kirillov [30] by simply replacing t by the root of unity in Eq.(2.20). This is permissible if and only if the integral of two-form $\int_\gamma \omega_x$ taken over any two dimensional cycle γ on coadjoint orbit $O^*(x)$ is integer[15]. But this is exactly the condition which makes the Kähler complex structure that of the Hodge type [10].

3.2 Construction of the moment map using methods of linear programming

In this subsection we are not going to employ the definition of moment mapping used in symplectic geometry [5][16]. Instead, we shall rely heavily on works by Atiyah [43,44] with only slightest improvement coming from noticed connections with linear programming not mentioned in his papers and in literature on symplectic geometry. In our opinion, such connection is helpful for better physical understanding of mathematical methods presented in this paper.

Using results and terminology of Section 2.1. and Appendix A (part c)) we call a subset of $\mathbf{R}^n$ a polyhedron $\mathcal{P}$ if there exist $m \times n$ matrix A (with $m < n$) and a vector $b \in \mathbf{R}^m$ such that

$$\mathcal{P} = \{\mathbf{x} \in \mathbf{R}^n \mid Ax \leq b\}. \tag{3.7}$$

Since each component of the inequalities $Ax \leq b$ determines the half space while the equality $Ax = b$ -the underlying hyperplane, the polyhedron is an intersection of finitely many halfspaces. The problem of linear programming can be formulated as follows [45] : for linear functional $\tilde{\mathcal{H}}[\mathbf{x}]$=$\mathbf{c}^{\mathrm{T}} \cdot \mathbf{x}$ find max $\tilde{\mathcal{H}}[\mathbf{x}]$ on $\mathcal{P}$ provided that the vector $\mathbf{c}$ is assigned. It should be noted that this problem is just one of many related. It was selected only because of its immediate relevance. Its relevance comes from the fact that the extremum of $\tilde{\mathcal{H}}[\mathbf{x}]$ is achieved at least at one of the vertices of $\mathcal{P}$ .The proof of this we omit since it can be found in any standard textbook on linear programming, e.g. see [46] and references therein. This result does not require the polyhedron to be centrally symmetric[17]. Only convexity of polyhedron is of importance.

To connect the above optimization problem with the results of this paper we constrain $\mathbf{x}$ variables to integers, i.e. to $\mathbf{Z}^n$. Such restriction is known in literature as *integer linear programming.* In our case, it is equivalent to considering symplectic manifolds of Hodge-type (e.g. read page 11 of Atiyah's paper, Ref.[43]). As a warm up exercise which, in part, we shall need in the next section as well, following Fulton [26], consider a deformation retract of complex projective space $\mathbf{CP}^n$ which is the simplest possible toric variety [24]. Such retraction is achieved by using the map :

$$\tau : \ \mathbf{CP}^n \to \mathbf{P}^n_{\geq} = \mathbf{R}^{n+1}_{\geq} \setminus \{0\}/\mathbf{R}^+$$

[15] An important example of such quantization is discussed in the next subsection.

[16] Although, essentaillly, we are going to use the same thing.

[17] This fact is important for possible potential extension of the results of this work which do require central symmetry.

given explicitly by

$$\tau :\ (z_0,...,z_n) \mapsto \frac{1}{\sum_i |z_i|}(|z_0|,...,|z_n|) = (t_0,...,t_n)\ ,\ t_i \geq 0 \tag{3.8}$$

so that, naturally, the mapping τ is onto the standard n-simplex : $t_i \geq 0$, $t_0 + ... + t_n = 1$. To bring physics to this example, consider the Hamiltonian for harmonic oscillator. In the appropriate system of units we can write it as $\mathcal{H} = m(p^2+q^2)$. More generally, for the set of oscillators, i.e. for the "truncated" bosonic string, we have: $\mathcal{H}[\mathbf{z}] = \sum_i m_i |z_i|^2$, where, following Atiyah [43], we have introduced complex z_j variables via $z_j = p_j + iq_j$. Let now such Hamiltonian system (string) possess finite fixed energy $\mathcal{E}$. Then, we obtain:

$$\mathcal{H}[\mathbf{z}] = \sum\nolimits_{i=0}^{n} m_i |z_i|^2 = \mathcal{E}. \tag{3.9}$$

It is not difficult to realize that the above equation actually represents some surface in $\mathbf{CP}^n$ since the points z_j can be identified with points $e^{i\theta}z_j$ in Eq.(3.9) while keeping the above expression form- invariant. This observation is helpful in the rest of this paper. As before, we can map such model living in $\mathbf{CP}^n$ back into simplex in an obvious way. We can do better this time, however, since $\mathbf{CP}^n$ is the simplest case of toric variety [23]. If we let z_j to "live" in such variety it will be affected by the torus action as it was discussed in Section 2.4.This means that the masses in Eq.(3.9) will change and accordingly the energy. Only if we constrain the torus action to the roots of unity ξ^j will the energy be conserved. This provides another justification for their use (e.g. see earlier discussion next to Eq.(2.17)). Hence, from now on we are going to work with this case only. If this is the case, then $|z_i|^2$ is just some positive number and *the essence of the moment map lies exactly in such identification.* Hence, we obtain

$$\tilde{\mathcal{H}}[\mathbf{x}] = \sum\nolimits_{i=0}^{n} m_i x_i, \tag{3.10}$$

where we had removed the energy constraint for a moment thus making $\tilde{\mathcal{H}}[\mathbf{x}]$ to coincide with earlier defined linear functional to be optimized. Now we have to find the convex polyhedron on which such functional is going to be optimized. Thanks to works by Atiyah [43,44] and Guillemin and Sternberg [47], this task is completed. Naturally, the polyhedron emerges as intersection of images of the critical points (i.e. those for which the two-form $\omega_x = 0$) of the moment map [48]. Then, the theorem of linear programing stated earlier guarantees that $\tilde{\mathcal{H}}[\mathbf{x}]$ achieves its maximum at least at some of its vertices. Delzant [49] had demonstrated that this is the case without use of linear programming language.

It is helpful to demonstrate the essence of above arguments on the simplest but important example discussed originally by Frankel [50] and later elaborated in [51] . Consider a two sphere S^2 of unit radius, i.e. $x^2 + y^2 + z^2 = 1$ and parametrize this sphere using coordinates $x = \sqrt{1-z^2}\cos\phi$, $y = \sqrt{1-z^2}\sin\phi$, $z = z$.The Hamiltonian for the free particle "living" on such sphere is given by $\mathcal{H}[z] = m\left(1-|z|^2\right)$ so that equations of motion produce circles of latitude.

These circles become (critical) points of equilibria at the north and south pole of the sphere, i.e. for $z = \pm 1$. Under such circumstances our polyhedron is the segment [-1,1] and its vertices are located at ± 1 (to be compared with Section1.1.). The moment map $\mathcal{H}[x]$=$m\,(1-x)$ acquires its maximum at $x = 1$ and the value $x = 1$ corresponds to two polyhedral vertices located at 1 and -1 respectively. This doubling feature was noticed and discussed in detail by Delzant [49] whose work contains all needed proofs. These can be considered as elaborations on much earlier results by Frankel [50][18]. The circles on the sphere represent the torus action (e.g. read the discussion following Eq.(3.9)) so that dimension of the circle is half of that of the sphere. This happens to be a general trend : the dimension of the Cartan subalgebra (more accurately, the normalizer of maximal torus) is half of the dimension of the symplectic manifold $\mathcal{M}_x$ [48,50]. Incidentally, the integral of the symplectic two- form ω_x over S^2 is found to be 2 [51] so that the complex structure on the sphere is that of Hodge type. Generalization of this example to the multiparticle case can be found also in the same reference and will be used in Section 5.

The results discussed thus far although establish connection between the singularities of symplectic manifolds and polyhedra do not allow to discuss fine details which allow to distinguish between different polyhedra. Fortunately, this has been to a large degree accomplished [39,53]. Such task is equivalent to classification of all finite dimensional exactly integrable systems. In the next sections we shall argue that such information is also useful for classification of infinite dimensional exactly integrable systems of conformal field theories. In this case the situation is analogous to that encountered in solid state physics where free electrons with well defined mass end energy in vacuum acquire effective mass and multitude of energies when they live in periodic solids [36].

4 New Veneziano-like amplitudes from old Fermat (hyper) surfaces

4.1 Brief review of the Veneziano amplitudes

In 1968 Veneziano [54] postulated 4-particle scattering amplitude $A(s,t,u)$ given (up to a common constant factor) by

$$A(s,t,u) = V(s,t) + V(s,u) + V(t,u), \tag{4.1}$$

where

$$V(s,t) = \int_0^1 x^{-\alpha(s)-1}(1-x)^{-\alpha(t)-1}dx \equiv B(-\alpha(s), -\alpha(t)) \tag{4.2}$$

[18]E.g. read especially page 4 of Frankel's paper. His results, in turn, were largely influenced by still much earlier seminal paper by Hopf and Samelson [52].

is Euler's beta function and $\alpha(x)$ is the Regge trajectory usually written as $\alpha(x) = \alpha(0) + \alpha' x$ with $\alpha(0)$ and α' being the Regge slope and the intercept, respectively. In the case of space-time metric with signature $\{-,+,+,+\}$ the Mandelstam variables s, t and u entering the Regge trajectory are defined by [55]

$$\begin{aligned} s &= -(p_1+p_2)^2, \\ t &= -(p_2+p_3)^2, \\ u &= -(p_3+p_1)^2. \end{aligned} \tag{4.3}$$

The 4-momenta p_i are constrained by the energy-momentum conservation law leading to relation between the Mandelstam variables:

$$s+t+u = \sum_{i=1}^{4} m_i^2. \tag{4.4}$$

Veneziano [54] noticed[19] that to fit experimental data the Regge trajectories should obey the constraint

$$\alpha(s) + \alpha(t) + \alpha(u) = -1 \tag{4.5}$$

consistent with Eq.(4.4) in view of definition of $\alpha(s)$[20]. He also noticed that the amplitude $A(s,t,u)$ can be equivalently rewritten with help of this constraint as follows

$$A(s,t,u) = \Gamma(-\alpha(s))\Gamma(-\alpha(t))\Gamma(-\alpha(u))[\sin\pi(-\alpha(s)) + \sin\pi(-\alpha(t)) + \sin\pi(-\alpha(u))]. \tag{4.6}$$

The Veneziano amplitude looks strikingly similar to that suggested a bit later by Virasoro [57]. The latter (up to a constant) is given by

$$\bar{A}(s,t,u) = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b)\Gamma(b+c)\Gamma(c+a)} \tag{4.7}$$

with parameters $a = -\frac{1}{2}\alpha(s)$,etc. also subjected to the constraint:

$$\frac{1}{2}\left(\alpha(s) + \alpha(t) + \alpha(u)\right) = -1. \tag{4.8}$$

Use of the formulas

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin\pi x} \tag{4.9a}$$

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[19] To get our Eq.(4.5) from Eq.7 of Veneziano paper, it is sufficient to notice that his $1-\alpha(s)$ corresponds to ours -$\alpha(s)$.

[20] The Veneziano condition $\alpha(s)+\alpha(t)+\alpha(u) = -1$ can be rewritten in the form consistent with that considered earlier in the text. Indeed, let m, n, l be some integers such that $\alpha(s)m + \alpha(t)n + \alpha(u)l = 0$, then by adding this equation to that written above we obtain $\alpha(s)\tilde{m} + \alpha(t)\tilde{n} + \alpha(u)\tilde{l} = -1$,or, even more generally, $\alpha(s)\tilde{m} + \alpha(t)\tilde{n} + \alpha(u)\tilde{l} + \tilde{k}\cdot 1 = 0$.Both types of the equations had been extensively studied in the book by Stanley [56].

and

$$4 \sin x \sin y \sin z = \sin(x+y-z) + \sin(y+z-x) + \sin(z+x-y) - \sin(x+y+z) \tag{4.9b}$$

permits us to rewrite Eq.(4.7) in the alternative form (up to unimportant constant):

$$\begin{aligned} \bar{A}(s,t,u) &= [\Gamma(-\frac{1}{2}\alpha(s))\Gamma(-\frac{1}{2}\alpha(t))\Gamma(-\frac{1}{2}\alpha(u))]^2 \times \\ & [\sin \pi(-\frac{1}{2}\alpha(s)) + \sin \pi(-\frac{1}{2}\alpha(t)) + \sin \pi(-\frac{1}{2}\alpha(u))]. \end{aligned} \tag{4.10}$$

Although these two amplitudes look deceptively similar, mathematically, they are markedly different. Indeed, by using Eq.(4.6) conveniently rewritten as

$$A(a,b,c) = \Gamma(a)\Gamma(b)\Gamma(c)[\sin \pi a + \sin \pi b + \sin \pi c] \tag{4.11}$$

and exploiting the identity

$$\cos \frac{\pi z}{2} = \frac{\pi^z}{2^{1-z}} \frac{1}{\Gamma(z)} \frac{\zeta(1-z)}{\zeta(z)}$$

after some trigonometric calculations the following result is obtained:

$$A(a,b,c) = \frac{\zeta(1-a)}{\zeta(a)} \frac{\zeta(1-b)}{\zeta(b)} \frac{\zeta(1-c)}{\zeta(c)}, \tag{4.12}$$

provided that

$$a + b + c = 1. \tag{4.13}$$

For the Virasoro amplitude, apparently, no result like Eq.(4.12) can be obtained. As the rest of this section demonstrates, the differences between the Veneziano and the Virasoro amplitudes are much more profound. The result, Eq.(4.12), is also remarkable in the sense that it allows us to interpret the Veneziano amplitude from the point of view of number theory, the theory of dynamical systems, etc. Ref. [2] contains additional details of such interpretation. To our knowledge, no such interpretation is possible for the Virasoro amplitudes. For this and other reasons described below and in Appendix B we shall treat only Veneziano and Veneziano-like amplitudes in this paper.

4.2 From Veneziano amplitudes to hypergeometric functions and back

The fact that the hypergeometric functions are the simplest solutions of the Knizhnik-Zamolodchikov equations of CFT is well documented [58]. The connection between these functions and the toric varieties had been also developed in papers by Gelfand, Kapranov and Zelevinsky [59] (GKZ). Hence, we see no need in duplication of their results in this work. Instead, we would like to discuss other aspects of hypergeometric functions and their connections with the

Veneziano amplitudes emphasizing similarities and differences between strings and CFT.

For reader's convenience, we begin by introducing some standard notations. In particular, let

$$(a, n) = a(a+1)(a+2)\cdots(a+n-1)$$

and, more generally, $(a) = (a_1, ..., a_p)$ and $(c) = (c_1, ..., c_q)$. Then with help of these notations the p, q-type hypergeometric function is written as

$$_pF_q[(a);(c);x] = \sum_{n=0}^{\infty} \frac{(a_1,n)\cdots(a_p,n)}{(c_1,n)\cdots(c_q,n)} \frac{x^n}{n!}. \tag{4.14}$$

In particular,the hypergeometric function in the form known to Gauss is just $_2F_1 = F[a,b;c;x]$. Practically all elementary functions and almost all special functions can be obtained as special cases of the hypergeometric function just defined [60].

We are interested in connections between the hypergeometric functions and the Schwarz-Christoffel (S-C) mapping problem. The essence of this problem lies in finding a function $\varphi(\zeta) = z$ which maps the upper half plane $\operatorname{Im}\zeta > 0$ (or, which is equivalent, the unit circle) into the exterior of the $n-$sided polygon located on the Riemann sphere considered as one dimensional complex projective space $\mathbf{CP}^1$ (i.e. $z \in \mathbf{CP}^1$). Traditionally, the pre images $a_1, ..., a_n$ of the polygon vertices located at points $b_1, ..., b_n$ in $\mathbf{CP}^1$ are placed onto x axis of ζ-plane so that $\varphi(a_i) = b_i$, $i = 1-n$. Let the interior angles of the polygon be $\pi\alpha_1, ..., \pi\alpha_n$ respectively. Then the exterior angles μ_i are defined trough relations $\pi\alpha_i + \pi\mu_i = \pi$, $i = 1-n$. The exterior angles are subject to the constraint: $\sum_{i=1}^n \mu_i = 2$. The above data allow us to write for the S-C mapping function the following known expression:

$$\varphi(\zeta) = C\int_0^{\zeta}(t-a_1)^{-\mu_1}\cdots(t-a_n)^{-\mu_n} + C'. \tag{4.15}$$

If one of the points, say a_n, is located at infinity, it can be shown that in the resulting formula for mapping the last term under integral can be deleted.

Consider now the simplest but relevant example of mapping of the upper half plane into triangle with angles α, β and γ subject to Euclidean constraint: $\alpha + \beta + \gamma = 1$. Let, furthermore, $a_1 = 0, a_2 = 1$ and $a_3 = \infty$. Using Eq.(4.15) (with $C = 1$) we obtain for the length c of the side of the triangle:

$$c = \int_0^1 \left|\frac{d\varphi(\zeta)}{d\zeta}d\zeta\right| = \int_0^1 z^{\alpha-1}(1-z)^{\beta-1} = B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(1-\gamma)}. \tag{4.16}$$

Naturally, two other sides can be determined the same way. Much more efficient is to use the familiar elementary trigonometry relation

$$\frac{c}{\sin\pi\gamma} = \frac{b}{\sin\pi\beta} = \frac{a}{\sin\pi\alpha}.$$

Then, using Eq.(4.9a) we obtain for the sides the following results : $c = \frac{1}{\pi}\sin\pi\gamma\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma); b = \frac{1}{\pi}\sin\pi\beta\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma); a = \frac{1}{\pi}\sin\pi\alpha\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)$. The perimeter length $\mathcal{L} = a + b + c$ of the triangle is just the full Veneziano amplitude, Eq.(4.11). As is well known [61], the conformal mapping with Euclidean constraint $\alpha+\beta+\gamma = 1$ can be performed only for 3 sets of *fixed* angles. Another 4 sets of angles belong to the spherical case: $\alpha + \beta + \gamma > 1$, while the countable infinity of angle sets exist for the hyperbolic case: $\alpha+\beta+\gamma < 1$. Hence, the associated with such mappings Fuchsian-type equations used in some string theory formulations will not be helpful in deriving the Veneziano amplitudes. These equations are useful however in the CFT as is well known [58].

Fortunately, there is alternative formalism. It allows us to treat both string and CFT from the same mathematical standpoint. Although this formalism is discussed in some detail in previous sections there is still need to fill out some gaps before we can actually use it.

It is well documented that the bosonic string theory had emerged as an attempt at multidimensional generalization of Euler's beta function [62]. Analogous development also took place in the theory of hypergeometric functions where it was performed along two related lines. To illustrate the key ideas, following Mostow and Deligne [63] consider the standard hypergeometric function which, up to a constant[21], is given by

$$F[a, b; c; x] \doteq \int_1^\infty u^{a-c}(u-1)^{c-b-1}(u-x)^{-a}du. \tag{4.17}$$

The multidimensional (multivariable) analogue of the above function according to Picard (in notations of Mostow and Deligne) is given by

$$F[x_2, ..., x_{n+1}] = \int_1^\infty u^{-\mu_0}(u-1)^{-\mu_1}\prod_{i=2}^{n+1}(u-x_i)^{-\mu_i}du \tag{4.18}$$

provided that $x_0 = 0, x_1 = 1$ and, as before, $\sum_{i=0}^{n}\mu_i = 2$. At the same time, using alternative representation of $F[a, b; c; x]$ given by

$$F[a, b; c; x] \doteq \int_0^1 z^{b-1}(1-z)^{c-b-1}(1-zx)^{-a}dz \tag{4.19}$$

one obtains as well the following multidimensional generalization:

$$F[\alpha, \beta, \beta', \gamma; x, y] \doteq \int\int_{\substack{u\geq 0, v\geq 0\\ u+v\leq 1}} u^{\beta-1}v^{\beta'-1}(1-u-v)^{\gamma-\beta-\beta'-1}(1-ux)^{-\alpha}(1-v\,\mathrm{y})^{-\alpha'}dudv \tag{4.20}$$

[21] To indicate this we use symbol $\doteq$.

This result was obtained by Horn already at the end of 19th century and was subsequently reanalyzed and extended by GKZ. Looking at the last expression one can design by analogy the multidimensional extension of the Euler's beta function. In view of Eq.(4.2), it is given by the following integral attributed to Dirichlet:

$$\mathcal{D}(x_1,...,x_{k+1}) = \int \underset{\substack{u_1\geq 0,...,u_k\geq 0 \\ u_1+\cdots+u_k\leq 1}}{\int} \mathrm{u}_1^{x_1-1}\,\mathrm{u}_2^{x_2-1}\,...\,\mathrm{u}_k^{x_k-1}(1-u_1-...-u_k)^{x_{k+1}-1}du_1...du_k. \tag{4.21}$$

This result is going to be reobtained and explicitly calculated below in this section (e.g. see Eq.s(4.57) and (5.36) below) thus leading to multiparticle generalization of the Veneziano and Veneziano-like amplitudes. Before doing so we need to reproduce some results from Bourbaki [7]. They will be helpful in the next section as well.

4.3 Selected exercises from Bourbaki(begining)

To accomplish of our tasks we need to work out some problems listed at the end of Chapter 5 paragraph 5 (problem set # 3) of Bourbaki [7]. Fortunately, answers to these problems to a large extent (but not completely!) can be extracted from the paper by Solomon [8]. In view of their crucial mathematical and physical importance we reproduce major relevant results in this subsection. Section 5 (Subsection 5.1.2.) contains the rest of relevant results from this exercise.

Let K be the field of characteristic zero (e.g.$\mathbf{C}$) and V be the vector space of finite dimension l over it. Let G be a subgroup of $GL_l(V)$ made of pseudo-reflections acting on V. Let q be the cardinality of G. Introduce now the symmetric $S(V)$ and exterior $E(V)$ algebra of V and look for invariants of pseudo-reflection groups made of $S(V)$ and $E(V)$. This task requires several steps. First, multiplication of polynomials leads to the notion of graded ring R[22]. For example, if the polynomial $P_i(x)$ of degree i belongs to the polynomial ring $\mathbf{F}[x]$ then the product $P_i(x)P_j(x) \in P_{i+j}(x) \in \mathbf{F}[x]$. A graded ring R is a ring with decomposition $R = \oplus_{j=\mathbf{Z}} R_j$ compatible with addition and multiplication. Next, for the vector space V if $x = x_1 \otimes \cdots \otimes x_s \in V^{\otimes_s}$and $y = y_1 \otimes \cdots \otimes y_t \in V^{\otimes_t}$, then the product $x \otimes y \in V^{\otimes_{s+t}}$.The multitude of such type of tensor products forms noncommutative associative algebra $T(V)$. Finally, the symmetric algebra $S(V)$ is defined by $S(V) = T(V)/I$, where the ideal I is made of $x \otimes y - y \otimes x$ (with both x and $y \in V$). In practical terms $S(V)$ is made of symmetric polynomials $\mathbf{F}[t_1,...,t_l]$ with $t_1,...,t_l$ being in one-to-one correspondence with basis elements of V (that is each of $t_i's$ is entering into $S(V)$ with power one). The exterior algebra $E(V)$ can be defined analogously now. For this we need to map the vector space V into the Grassmann algebra of V .This is done just like in physics when one goes from bosonic functions (belonging to $S(V)$) to fermionic functions (belonging to $E(V)$). In particular,

[22]Surely, once the definition of such ring is given, there is no need to use polynomilals. But in the present case this analogy is useful.

if $x \in V$ then, its image in the Grassmann algebra $\tilde{x}$ possess a familiar (to physicists) property : $\tilde{x}^2 = 0$. The graded two- sided ideal I can be defined now as

$$I = \{\tilde{x}^2 = 0 \mid \; x \to \tilde{x}; x \in V \} \tag{4.23}$$

so that $E(V) = T(V)/I$. To complicate matters a little bit consider a map $d : x \to dx$ for $x \in V$ and dx belonging to the Grassmann algebra. If $t_1, ..., t_l$ is the basis of V , then $dt_{i_1} \wedge \cdots \wedge dt_{i_k}$ is the basis of $E_k(V)$ with $0 \leq k \leq l$ and , accordingly, the graded algebra $E(V)$ admits the following decomposition: $E(V) = \oplus_{k=0}^{l} E_k(V)$. Next, we need to construct the invariants of pseudo-reflection group G made out of $S(V)$ and $E(V)$ and, most importantly, out of tensor product $S(V) \otimes E(V)$. Toward this goal we need to look if the action of the map $d : V \to E(V)$ extends to a differential map

$$d : S(V) \otimes E(V) \to S(V) \otimes E(V). \tag{4.24}$$

Clearly, $\forall x \in E(V)$ we have $d(x) = 0$.Therefore, $\forall x, y \in S(V) \otimes E(V)$ we can write $d(x, y) = d(x)y + xd(y)$. By combining these two results together we obtain

$$d : S_i(V) \otimes E_j(V) \to S_{i-1}(V) \otimes E_{j+1}(V) \tag{4.25}$$

, i.e. the differentiation is compatible with grading. Now we are ready to formulate the theorem by Solomon [8] which is of central importance for this work. It is formulated in the form stated in Bourbaki [7].

Theorem 2 (Solomon [8]) . *Let $P_1, ..., P_k$ be algebraically independent polynomial forms (made of symmetric combinations of $t_1, ..., t_k$) generating the ring $S(V)^G$ of invariants of G. Then, every invariant differential p-form $\omega^{(p)}$ may be written uniquely as a sum*

$$\omega^{(p)} = \sum_{i_1 < \ldots < i_p} c_{i_1 \ldots i_p} dP_{i_1} \cdots dP_{i_p} \quad ; 1 \leq p \leq k \tag{4.26}$$

with $c_{i_1 \ldots i_p} \in S(V)^G$.Moreover, actually, the differential forms $\Omega^{(p)} = dP_{i_1} \wedge \cdots \wedge dP_{i_p}$ with $1 \leq p \leq k$ generate the entire algebra of $G-$ invariants of $S(V) \otimes E(V)$.

Corollary 1. Let $t_1, ..., t_k$ be the basis of V . Furthermore, let $S(V) = \mathbf{F}[t_1, ..., t_k]$ be its algebra of symmetric polynomials and $S(V)^G = \mathbf{F}[P_1, ..., P_l]$ its finite algebra of $G-$invariants[23]. Then, since $dP_i = \sum\limits_j \frac{\partial P_i}{\partial t_j} dt_j$ we have

$$dP_1 \wedge \cdots \wedge dP_k = J(dt_1 \wedge \cdots \wedge dt_k) \tag{4.27}$$

where, up to a constant factor $c \in K$, the Jacobian J is given by $J = c\Omega$ with

$$\Omega = \prod_{i=1}^{\nu} L_i^{c_i - 1} \tag{4.28}$$

[23] The fact that the number of polynomial forms P_i is equal to the rank l of G is not a trivial fact. The proof can be found in [64] p.128. Incidentally, this proof implies immediately the result, Eq.(4.28), given below.

In this equation L_i is *linear* form defining $i-th$ reflecting hyperplane H_i (it is assumed that the set of $H_1, ..., H_\nu$ reflecting hyperplanes is associated with G) , i.e. $H_i = \{\alpha \in V \mid L_i(\alpha) = 0\}$ as defined in the Appendix A. The set of all elements of G fixing H_i pointwise forms a cyclic subgroup of order c_i generated by pseudoreflections[24].

Remark 4. The result given by Eq.(4.28) as well as the proportionality : $J = c\Omega$, can be found in the paper by Stanley [65]. It can be also found in much earlier paper by Solomon [8] where it is attributed to Steinberg and Shephard and Todd. Stanley's paper contains some details missing in earlier papers however.

Remark 5. Using Theorem 2 by Solomon, Ginzburg [9] proved the following

Theorem 3.(Ginzburg [9], page 358) *Let $\omega_x(\xi_1, \xi_2)$ be a symplectic (Kirillov-Kostant) two- form defined in Section 3.1, let $\Omega^N = \omega_x^N$ be its N-th exterior power -the volume form, with N being the number of positive roots of the associated Weyl-Coxeter reflection group, then*

$$* \left(\Omega^N\right) = const \cdot dP_1 \wedge \cdots \wedge dP_k$$

where the star $*$ denotes the standard $Hodge-type$ star operator .

Corollary 2. As it was stated in Section 3.1., every nonsingular algebraic variety in projective space is symplectic. The symplectic structure gives raise to the complex Kähler structure which, in turn, is of Hodge-type for Kirillov -type symplectic manifolds.

Remark 6. In seminal work, Ref. [66], Atiyah and Bott had argued that $\omega^{(p)}$ can be used as basis of the equivariant cohomology ring. This result will be used in Section 5. We refer our readers to the monograph [67] by Guillemin and Sternberg where all concepts of equivariant cohomology are pedagogically explained.

Obtained results are sufficient for reobtaining the Dirichlet integral, Eq.(4.21), and, more generally, the multiparticle Veneziano-like amplitudes. They also provide needed mathematical background for adequate physical interpretation. The next subsection makes all these statements explicit.

4.4 Veneziano amplitudes from Fermat hypersurfaces

4.4.1 General considerations

Let us begin with observation that all integrals of Section 4.2. can be reobtained with help of theorem by Solomon (Theorem 2). Indeed, let V be the complex affine space of dimension l and let $L_i(v), v \in V$ be the linear form defining $i-$th hyperplane H_i, i.e.

$$\mathrm{H}_i = \{v \in V \mid L_i(v) = 0\} \ , \ i = 1, ..., \ l. \tag{4.29}$$

[24] E.g. read Appedix A part c).

As in the theory of linear programming discussed in Section 3.2., the set of linear equations given above defines a polyhedron $\mathcal{P}$. In view of Eq.s(4.26)-(4.29), consider an integral I of the type

$$I=\int\limits_{\mathcal{P}} c_{i_1\ldots i_p} dP_{i_1}\wedge\cdots\wedge dP_{i_p} \quad ; 1\leq p\leq l. \tag{4.30}$$

This is a typical integral of general hypergeometric type. All integrals of Section 4.2. are of this type [60]. They can be obtained as solutions of the associated with them systems of differential equations of the Picard-Fuchs (P-F) type [60] also used in the mirror symmetry calculations [68].

In this work, we shall discuss another, more direct, option which avoid use of the P-F type equations. This enable us to find physical applications of mathematical results not discussed to our knowledge in mathematical physics literature.

We begin with the simplest example borrowed from the paper by Griffiths [69]. His paper begins with calculation of the following integral for the period π

$$\pi(\lambda)=\oint_{\Gamma}\frac{dz}{z(z-\lambda)} \tag{4.31}$$

along the closed contour Γ in the complex z-plane. Since this integral depends upon parameter λ the period $\pi(\lambda)$ is some function of λ. It can be determined by straightforward differentiation of $\pi(\lambda)$ with respect to λ thus leading to the desired differential equation

$$\lambda\pi'(\lambda)+\pi(\lambda)=0 \tag{4.32}$$

enabling us to calculate $\pi(\lambda)$. This simple result can be vastly generalized to cover the case of period integrals of the type

$$\Pi(\lambda)=\oint\limits_{\Gamma}\frac{P\left(z_1,...,z_n\right)}{Q\left(z_1,...,z_n\right)}dz_1\wedge dz_2\cdots\wedge dz_n. \tag{4.33}$$

The equation $Q\left(z_1,...,z_n\right)=0$ determines algebraic variety. It may conatain a parameter (or parameters) λ so that the polar locus of values of $z's$ satisfying equation $Q=0$ depends upon this parameter(s). By analogy with Eq.(4.32) it is possible to obtain a set of differential equations of P-F type. This was demonstrated originally by Manin [70]. In this work we are not going to develop this line of research however. Instead, following Griffiths [69], we want to analyze in some detail the nature of the expression under the sign of integral in Eq.(4.33).

If $\mathbf{x}=(x_{0,}...,x_n)$ are homogenous coordinates of a point in *projective space* and $z=(z_1,...,z_n)$ are the associated coordinates of the point in the *affine space* where $z_i=x_i/x_0$, then, the rational $n-$form ω is given in the *affine* space by

$$\omega=\frac{P\left(z_1,...,z_n\right)}{Q\left(z_1,...,z_n\right)}dz_1\wedge dz_2\cdots\wedge dz_n \tag{4.34}$$

with rational function P/Q being a quotient of two homogenous polynomials of the *same* degree. Upon substitution $z_i = x_i/x_0$ the form $dz_1 \wedge dz_2 \cdots \wedge dz_n$ changes to

$$dz_1 \wedge dz_2 \cdots \wedge dz_n = (x_0)^{-(n+1)} \sum_{i=0}^{n} (-1)^i x_i dx_0 \wedge ... \wedge d\hat{x}_i \wedge ... \wedge dx_n$$

where the hat on the top of x_i means that it is excluded from the product. It is convenient now to define the form ω_0 via

$$\omega_0 := \sum_{i=0}^{n} (-1)^i x_i dx_0 \wedge ... \wedge d\hat{x}_i \wedge ... \wedge dx_n$$

so that in terms of *projective space* coordinates the form ω can be rewritten as

$$\omega = \frac{p(\mathbf{x})}{q(\mathbf{x})} \omega_0 \tag{4.35}$$

where $p(\mathbf{x}) = P(\mathbf{x})$ and $q(\mathbf{x}) = Q(\mathbf{x}) x_0^{n+1}$ or, in more symmetric form, $q(\mathbf{x}) = Q(\mathbf{x}) x_0 \cdots x_n$. In this case the degree of denominator of the rational function p/q is that of numerator $+(n+1)$.This is the result of Corollary 2.11 of Griffith's paper [69].Conversely, each homogenous differential form ω in projective space can be written in affine space upon substitution : $x_0 = 1$ and $x_i = z_i, i \neq 0$.

We would like to take advantage of this fact now. To this purpose, as an example, we would like to study the period integrals associated with equation describing Fermat hypersurface in complex projective space

$$\mathcal{F}(N) : x_0^N + \cdots + x_n^N + x_{n+1}^N = 0. \tag{4.36}$$

We would like now to combine Griffith's Corollary 2.11. with Solomon's Theorem 2. That is to say we would like to consider the set of independent linear forms $x_i^{<c_i>}$, $i = 0 - (n+1)$,where $< c_i >$ denotes representative of c_i in $\mathbf{Z}$ such that $1 \leq < c_i > \leq N - 1$[25] and $x_i = \mathrm{H}_i$, Eq.(4.29). They belong to the set of hyperplanes in $\mathbf{C}^{n+1}$ whose complement is complex algebraic torus T (Definition 9, Section 2.4). We want to consider the form ω living at the intersection of T with $\mathcal{F}$. To this purpose introduce $< c >$ as

$$< c > = \frac{1}{N} \sum_i < c_i > \tag{4.37}$$

where the numbers c_i have been defined earlier, after Eq.(4.28). They belong to the set $X(S^1)$:

$$X(S^1) = \{\bar{c} \in (\mathbf{Z}/N\mathbf{Z})^{n+2} \equiv (\mathbf{Z}/N\mathbf{Z}) \times \cdots \times (\mathbf{Z}/N\mathbf{Z}) \mid \bar{c} = (c_0, ..., c_{n+1}), \sum_i c_i = 0 \bmod N\} \tag{4.38}$$

[25]These limits for $< c_i >$ are in accord with Gross [71], page 198. Subsequently, they will be changed to $1 \leq < c_i > \leq N$, e.g. see Eq.(4.60) and the discussion around it.

fixing hyperlanes H_i pointwise. The condition $\sum_i c_i = 0 \bmod N$ is in accord with Eq.(2.22) as required. Its role and true meaning is illustrated by using the Fermat hypersurface $\mathcal{F}$ as an example. By combining Solomon's Theorem 2 with Griffiths Corollary 2.11. the form ω,Eq.(4.35), is given by

$$\omega = \frac{x_0^{<c_0>-1} \cdots x_{n+1}^{<c_{n+1}>-1}}{\left(x_0^N + \cdots + x_n^N + x_{n+1}^N\right)^{<c>}}\ \omega_o. \tag{4.39}$$

By design, it satisfies all the requirements just described.

4.4.2 The 4-particle Veneziano-like amplitude

Using Eq.(4.39), let us consider the simplest but important case : $n = 1$. It is relevant for calculation of 4-particle Veneziano-like amplitude. Converting ω into affine form according to Griffiths prescription and taking into account Solomon's Theorem 2 we obtain the following result for the period integral:

$$I_{aff} = \oint_\Gamma \frac{1}{\left(x_1^N + x_2^N \mp 1\right)} dx_1^{<c_1>} \wedge dx_2^{<c_2>}. \tag{4.40}$$

The $\pm$ sign in the denominator requires some explanation. Indeed, let us for a moment restore the projective form of this integral. We obtain the following integral:

$$I_{proj} = \oint_\Gamma \frac{z_1^{<c_1>} z_2^{<c_2>} z_0^{<c_0>}}{\left(z_1^N + z_2^N \pm z_0^N\right)}\left(\frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} - \frac{dz_0}{z_0} \wedge \frac{dz_2}{z_2} + \frac{dz_0}{z_0} \wedge \frac{dz_1}{z_1}\right). \tag{4.40a}$$

By construction, it is manifestly symmetric with respect to permutation of its arguments. In addition, the integrand is manifestly torus action invariant in the sense of Eq.s (2.16),(2.22). If we use $\xi^j = \exp(\pm i\frac{2\pi j}{N})$ with $1 \leq j \leq N-1$ the numerator of the integrand above acquires the factor $\exp\{i\frac{2\pi}{N}(< c_1 > j + < c_2 > k + < c_0 > l)\}$. Eq.(2.22) when combined with Griffiths Corollary 2.11 imposes the constraint

$$< c_1 > j + < c_2 > k + < c_0 > l = N. \tag{4.41a}$$

We shall call it the "Veneziano condition" while Eq.(4.41b) (below) we shall call the "Shapiro-Virasoro" condition.[26] It is needed to make the entire integrand torus action invariant. Transition from projective to affine space brakes the permutational symmetry firstly because of selecting, say, z_0(and requiring it to be one) and, secondly, by possibly switching sign in front of z_0. The permutational symmetry can be restored in the style of Veneziano, e.g. see Eq.(4.1). The problem of switching the sign in front of z_0 can be treated similarly but

[26]These names are given by analogy with the existing terminilogy for open (Veneziano) and closed (Shapiro-Virasoro) bosonic strings. Clearly, in the present context they emerge for reasons different from that used in conventional formulations.

requires extra care. This is so because instead of the factor ξ^j used above we could use ε^j as well. Here $\varepsilon = \exp(\pm i\frac{\pi}{N})$. Use of such factor makes the integral I_{proj} also torus action invariant. For this case the condition, Eq.(4.41a), has to be changed to

$$< c_1 > j + < c_2 > k + < c_0 > l = 2N \tag{4.41b}$$

in accord with Lemma 1 of Gross [71]. By such change we are in apparent disagreement with Corollary 2.11. We write "apparent" because, fortunately, there is a way to reconcile Corollary 2.11. by Griffiths with Lemma 1 by Gross. It will be discussed below. Already assuming that this is the case, we notice that there are at least two different classes of transformations leaving I_{proj} unchanged. When switching to the affine form these two classes are not equivalent: the first leads to differential forms of the first kind while the second-to the second kind. Both are living on the Jacobian variety $J(N)$ associated with the Fermat surface $\mathcal{F}(N) : z_1^N + z_2^N \pm 1 = 0$. It happens, that physically more relevant are the forms of the second kind. We would like to describe them now.

We begin by noticing that in switching from projective to affine space the following set of $3N$ points (at infinity) should be deleted from the Fermat curve $z_1^N + z_2^N + z_3^N = 0$.These are: $(\varepsilon\xi^j, 0, 1)$, $(0, \varepsilon\xi^j, 1), (\varepsilon^2\xi^j, \varepsilon\xi^j, 0)$ respectively. By assuming that this is the case and paramerizing: $z_1 = \varepsilon t_1^{\frac{1}{N}}$ and $z_2 = \varepsilon t_2^{\frac{1}{N}}$(by analogy with Eq.(3.8)), we obtain the simplex equation $t_1 + t_2 = 1$ as deformation retract for $\mathcal{F}(N)$. After this, Eq.(4.40) acquires the following form :

$$I_{aff} = \xi^{j<c_1>+k<c_2>}\frac{1}{N^2}\oint\limits_{\Gamma}\frac{\varepsilon^{<c_1>}t_1^{\frac{<c_1>}{N}}\varepsilon^{<c_2>}t_2^{\frac{<c_2>}{N}}}{(t_1+t_2-1)}\frac{dt_1}{t_1}\wedge\frac{dt_2}{t_2}. \tag{4.42}$$

The overall phase factor guarantees the linear independence of the above period integrals in view of the well-known result: $1 + \xi^r + \xi^{2r} + ... + \xi^{(N-1)r} = 0$. It will be omitted for brevity in the rest of our discussion.

To calculate I_{aff} we need to use generalization of method of residues for multidimensional complex integrals as developed by Leray [72] and discussed in physical context e.g. by Hwa and Teplitz [73] and others. From this reference we find that taking the residue can be achieved either by dividing the differential form in Eq.(4.42) by $ds = t_1 dt_1 + t_2 dt_2$ or, what is equivalent, by writing instead of Eq.(4.42) the following physically suggestive result

$$I_{aff} = \frac{1}{N^2}\oint\limits_{\Gamma}\varepsilon^{<c_1>}t_1^{\frac{<c_1>}{N}}\varepsilon^{<c_2>}t_2^{\frac{<c_2>}{N}}\frac{dt_1}{t_1}\wedge\frac{dt_2}{t_2}\delta(t_1+t_2-1) \tag{4.43}$$

to be discussed further in Section 5[27]. For the time being, taking into account that $t_2 = 1 - t_1$, after calculating the Leray residue we obtain,

$$I_{aff} = \frac{1}{N^2}\int\limits_0^1 \mathrm{u}^{\frac{<c_1>}{N}-1}(1-\mathrm{u})^{\frac{<c_2>}{N}-1}d\mathrm{u} = \frac{1}{N^2}B(a,b) \tag{4.44}$$

[27]Keeping in mind that $\delta(x) = \delta(-x)$ gives additional support in favour of $\varepsilon-$factors.

where $B(a,b)$ is Euler's beta function (as in Eq.(4.2)) with $a = \frac{<c_1>}{N}$ and $b = \frac{<c_2>}{N}$. The phase factors had been temporarily suppressed for the sake of comparison with the results of Rohrlich [71] (published as an Appendix to paper by Gross and also discussed in the book by Lang [74]). To make such comparison, we need to take into account the multivaluedness of the integrand above if it is considered in the standard complex plane. Referring our readers to Ch-r 5 of Lang's book [74] allows us to avoid rather long discussion about the available choices of integration contours. Proceeding in complete analogy with the case considered by Lang, we obtain the period $\Omega(a,b)$ of the differential form $\omega_{a,b}$ of the *second* kind living on $J(N)$:

$$\frac{\Omega(a,b)}{N} = \frac{1}{N}\int\limits_{\kappa} \omega_{a,b} = \frac{1}{N^2}(1-\varepsilon^{<c_1>})(1-\varepsilon^{<c_2>})B(a,b). \tag{4.45}$$

The Jacobian $J(N)$ is related to the Fermat curve $\mathcal{F}(N)$ considered as Riemann surface of genus $g = \frac{1}{2}(N-1)(N-2)$. Obtained result differs from that by Rohrlich only by phase factors : $\varepsilon' s$ instead of $\xi' s$. The number of such periods is determined by the inequalities of the type $1 \leq < c_i > \leq N-1$. In addition to the differential forms of the second kind there are also the differential forms of the *third* kind living on $\mathcal{F}(N)$.They can be easily obtained from that of the second kind by relaxing the condition $1 \leq < c_i > \leq N-1$ to $1 \leq < c_i > \leq N$, Lang [74], page 39. The differential forms of the second kind are associated with the de Rham cohomology classes $\mathrm{H}^1_{DR}(\ \mathcal{F}(N), \mathbf{C})$ (Gross [71], Lemma 1). The differentials forms of the first kind, discussed in the book by Lang [74], by design do not have any poles while the differentials of the second kind by design do not have residues. Only differentials of the third kind have poles of order ≤ 1 with nonvanishing residues and hence, are physically interesting. We shall be dealing mostly with differentials of the 2nd kind converting them eventually into that of the third kind. The differentials of the third kind are linearly independent from that of the first kind according to Lang [74].

Symmetrizing our result, Eq.(4.45), following Veneziano we obtain the 4-particle Veneziano-like amplitude

$$A(s,t,u) = V(s,t) + V(s,u) + V(t,u) \tag{4.46}$$

where, for example, upon analytical continuation $V(s,t)$ is given by

$$V(s,t) = (1-\exp(i\frac{\pi}{N}(-\alpha(s))(1-\exp(i\frac{\pi}{N}(-\alpha(t))B(\frac{-\alpha(s)}{N},\frac{-\alpha(t)}{N}), \tag{4.47}$$

provided that we had identified $< c_i >$ with $\alpha(i), etc.$ Naturally, in arriving at Eq.(4.47) we had extended the differential forms from that of the second kind to that of the third. The analytical properties of such designed Veneziano-like amplitudes are discussed in detail the subsection on multiparticle amplitudes (below) and in Appendix B.

4.4.3 Connection with CFT trough hypergeometric functions and the Kac–Moody-Bloch-Bragg condition

To complete our treatment of the 4-particle Veneziano amplitude several items still need to be discussed. In particular, we would like to compare the hypergeometric function, Eq.(4.19), with the beta function in the light of just obtained results. Taking into account that [60]

$$(1-zx)^{-a}=\sum_{n=0}^{\infty}\frac{(a,n)}{n!}(zx)^{n}$$

Eq.(4.19) can be rewritten as follows:

$$\begin{aligned} F[a,b;c;x] &\doteq \sum_{n=0}^{\infty}\frac{(a,n)}{n!}x^{n}\int_{0}^{1}z^{b+n-1}(1-z)^{c-b-1}dz \\ & \sum_{n=0}^{\infty}\frac{(a,n)}{n!}x^{n}B(b+n,c-b). \end{aligned} \tag{4.48}$$

This result is to be compared with Eq.(4.44).To this purpose it is convenient to rewrite Eq.(4.44) in the following more general form (up to a constant factor):

$$I(m,l)\doteq\int_{0}^{1}\mathrm{u}^{\frac{<c_1>-N+mN}{N}}(1-\mathrm{u})^{\frac{<c_2>-N+lN}{N}}d\mathrm{u}=B(a+m,b+l),$$

where $m,l=0,\pm1,\pm2,..$ It is clear, that the phase factors entering into Eq.(4.45) will either remain unchanged or will change sign upon such replacements. At the same time the Veneziano condition, Eq.(4.41), will change into

$$<c_0>+<c_1>+<c_2>=N+mN+lN+kN. \tag{4.49}$$

This result can be explained physically with help of some known facts from solid state physics, e.g. read Ref.[36]. To this purpose, using Sections 2.2 and 2.4 let us consider the result of torus action on the form ω, Eq.(4.39). If we demand this action to be invariant in accord with both Eq.(2.16) and Theorem 2 by Solomon, then we obtain,

$$\sum_{i}<c_i>m_i=0 \bmod N \tag{4.50}$$

with integers m_i having the same meaning as numbers $(l_j)_i$ in Eq.(2.9). Consider Eq.(4.50) for a special case of 4-particle Veneziano amplitude. Then, according to the footnote after Eq.(4.5) the Veneziano condition can be rewritten as

$$<c_0>m_0+<c_1>m_1+<c_2>m_2=0 \bmod N \tag{4.51}$$

to be compared with Eq.(2.22). But, in view of the Griffiths Corollary 2.11 the condition $\bmod N$ (or $\bmod 2N$) for the Veneziano amplitudes should actually

be replaced by N (or $2N$). At the same time for hypergeometric functions in view of Eq.s(4.49), (4.51), we should write instead

$$< c_0 > m_0 + < c_1 > m_1 + < c_2 > m_2 = mN + lN + kN. \tag{4.52}$$

Such condition is known in solid state physics as Bragg equation [36] . This equation plays the central role in determining crystal structure by X-ray diffraction. Lattice periodicity reflected in this equation affects kinematics of scattering processes for phonons and electrons in crystals. Under such conditions the concepts of particle energy and momentum loose their usual meaning and should be amended to account for lattice periodicity. The same type of amendments should be made when comparing elementary scattering processes in CFT against those in high energy physics. In view of Eq.(2.22) and results of Appendix A we shall call Eq (4.52) the *Kac-Moody-Bloch-Bragg (K–M-B-B) equation.*

4.4.4 The multiparticle Veneziano amplitudes and their analytic properties

By analogy with the 4-particle case, the Fermat variety $\mathcal{F}_{aff}\ (N)$ in the affine form in the multiparticle case is given by the following equation

$$\mathcal{F}_{aff}(N): \quad Y_1^N + \cdots + Y_{n+1}^N = 1, \ Y_i = x_i/x_0 \equiv z_i. \tag{4.53}$$

As before, use of parametrization $f : z_i = t_i^{\frac{1}{N}} \exp(\pm\frac{\pi i}{N})$ such that $\sum_i t_i = 1$ allows us to reduce the Fermat variety $\mathcal{F}_{aff}(N)$ to its deformation retract which is $n+1$ simplex Δ. I.e. $f(\ \mathcal{F}_{aff}(N)) = \Delta$, where $\Delta : \sum_i t_i = 1$. The period integrals of type given by Eq.(4.33) with ω form given by Eq.(4.39) after taking the Leray-type residue are reduced to the following standard form (up to a constant):

$$I \doteq \int\limits_{\Delta} t_1^{\frac{<c_1>}{N}-1} \cdots t_{n+1}^{\frac{<c_{n+1}>}{N}-1} dt_1 \wedge \cdots \wedge dt_n, \tag{4.54}$$

where, again, all phase factors had been suppressed temporarily. For $n = 1$ this integral coincides with that given in Eq.(4.44) (up to a constant) as required. As part of preparations for calculation of this integral for $n > 1$ let us have another look at the case $n = 1$ first where we have integrals of the type

$$I = \int\limits_0^1 dx x^{a-1}(1-x)^{b-1} = B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Alternatively, we can look at

$$\Gamma(a+b)I = \int\limits_0^\infty \int\limits_0^\infty dx_1 dx_2 x_1^{a-1} x_2^{b-1} \exp(-x_1 - x_2). \tag{4.55}$$

In the double integral on the r.h.s. let us consider change of variables: $x_1 = \hat{x}_1 t$, $x_2 = \hat{x}_2 t$ so that $x_1 + x_2 = t$ provided that $\hat{x}_1 + \hat{x}_2 = 1$. Taking t and $\hat{x}_1$ as

new variables and taking into account that the Jacobian of such transformation is one we obtain the following result,

$$\Gamma(a+b)I=\int\limits_0^\infty dt t^{a+b-1}\exp(-t)\int\limits_0^1 d\hat{x}_1\hat{x}_1^{a-1}(1-\hat{x}_1)^{b-1},$$

as expected. Going back to our original integral, Eq.(4.54) and introducing notations $a_i=\frac{<c_i>}{N}$ we obtain

$$\Gamma(\sum\nolimits_{n=1}^{n+1}a_i)I\doteq\int\limits_0^\infty\frac{dt}{t}t^{\sum_{i=1}^{n+1}a_i}\exp(-t)\int\limits_\Delta t_1^{a_1-1}\cdots t_{n+1}^{a_{n+1}-1}dt_1\wedge\cdots\wedge dt_n. \quad (4.56)$$

By analogy with the case $n=1$ we introduce new variables :$s_i=tt_i$. Naturally, we expect $\sum_{n=1}^{n+1}s_i=t$ since t_i variables are subject to the simplex constraint $\sum_{i=1}^{n+1}t_i=1$.With such replacements we obtain

$$\begin{aligned}\Gamma(\sum\nolimits_{n=1}^{n+1}a_i)I &= \int\limits_0^\infty\cdots\int\limits_0^\infty\exp(-\sum\nolimits_{n=1}^{n+1}s_i)s_1^{a_1}\cdots s_{n+1}^{a_{n+1}}\frac{ds_1}{s_1}\wedge\cdots\wedge\frac{ds_{n+1}}{s_{n+1}}\\ &= \Gamma(a_1)\cdots\Gamma(a_{n+1}).\end{aligned}$$

Using this result, finally, the $n-$particle contribution to the Veneziano amplitude is given by the following expression:

$$I\doteq\frac{\prod\limits_{i=1}^{n+1}\Gamma(a_i)}{\Gamma(\sum_{n=1}^{n+1}a_i)}. \quad (4.57)$$

Remark 7. Eq.(4.57) can be found in paper by Gross [71], page 206, where it is suggested (postulated) without derivation. Eq.(4.57) provides complete explicit calculation of the Dirichlet integral, Eq.(4.21), and, as such, can be found, for example, in the book by Edwards [75] published in 1922. Calculations similar to ours also can be found in lecture notes by Deligne [76]. We shall use some additional results from his notes below.

Our calculations are far from being complete however. To complete our calculations we need to introduce the appropriate phase factors. In addition, we need to discuss carefully the analytic continuation of just obtained expression for amplitude to negative values of parameters a_i . Fortunately, the phase factors can be reinstalled in complete analogy with the 4-particle case in view of the following straightforwardly verifiable identity

$$B(x,y)B(x+y,z)B(x+y+z,u)\cdots B(x+y+...+t,l)=\frac{\Gamma(x)\Gamma(y)\cdots\Gamma(l)}{\Gamma(x+y+...+l)} \quad (4.58)$$

Because of this identity, the multiphase problem is reduced to that we had considered already for the 4-particle case and, hence, can be considered as solved.

The analytic continuation problem connected with multiphase problem is much more delicate and requires longer explanations.

The first difficulty we encounter is related to the constraints imposed on $< c_i >$ factors discussed in connection with the 4-particle case, e.g. restriction :$1 \leq < c_i > \leq N-1$ (or $1 \leq < c_i > \leq N$). To resolve this difficulty we shall follow Deligne's lecture notes [76].We begin with Eq.(4.39). The Veneziano condition, Eq.(4.41a), extended to the multivariable case is written as

$$1 = < c > = \frac{1}{N} \sum_i < c_i > \tag{4.59}$$

whereas Griffiths Corollary 2.11. *does not* require this constraint to be imposed. To satisfy this corollary, it is sufficient only to require $m = < c >$ for some integer m to be specified below. Clearly, such requirement accordingly will change the total sum of exponents in the numerator of Eq.(4.39). In particular, for $m = 2$ we would reobtain back Eq.(4.41b). It should be noted at this point that Lemma 1 by Gross [71] although imposes such constraint but was proven *not* in connection with the period differential form, Eq.(4.39). This lemma implicitly assumes that the Leray residue *was taken already* and deals with the differential forms occurring as result of such operation. To avoid guessing in the present case we need to initiate our analysis again from Eq.(4.39) taking into account Corollary 2.11.

Following Deligne [76], let us discuss what happens if we replace $< c >$, Eq.(4.37), by $< -c >$. In view of definition of the bracket sign $< ... >$ we obtain,

$$< -c > = \frac{1}{N} \sum_i < -c_i > = \frac{1}{N} \sum_i < -c_i + N > = n + 2 - < c >, \tag{4.60}$$

where the factor $n+2$ comes from the sum $\sum_i 1$ and $< c >$ is the same as in Eq.(4.37) provided that $1 \leq < c_i > \leq N$. This result implies that the number m defined above can be only in the range

$$\frac{n+2}{N} \leq m \leq n+2. \tag{4.61}$$

Remark 8. Fermat variety $\mathcal{F}(N)$, Eq.(4.36), is of *Calabi-Yau* type if and only if $n+2 = N$ [77], page 531. Clearly, this requirement is equivalent to the Veneziano condition, Eq.(4.59), i.e. $m = 1$.

Remark 9. By not imposing this condition we can still get many interesting physically relevant results using Deligne's notes [76].We had encountered this already while using Eq.(4.41b). Clearly, this equation is reducible to Eq.(4.41a) anyway but earlier we obtained crucially physically important phase factor ε (instead of ξ) by working with Eq.(4.41b). It should be obvious by now that m is responsible for change in phase factors: from ξ (for $m = 1$)-to ε (for $m = 2$)-to $\hat{\varepsilon}_m = \exp(i\frac{2\pi}{mN})$ (for $m > 2$). Physical significance of these phase factors is discussed in the Appendix B.

To extend these results we need to introduce several new notations now. Let $V_{\mathbf{C}}$ be finite dimensional vector space over $\mathbf{C}$. A $\mathbf{C}$-rational Hodge structure of weight n on V is a decomposition $V_{\mathbf{C}} = \bigoplus_{p+q=n} V^{p,q}$ such that $\bar{V}^{p,q} = V^{q,p}$ We extend the definition of the torus action given by Eq.s(2.14),(2.15) in order to accommodate the complex conjugation : $(t, V^{p,q})$=$t^{-p}\bar{t}^{-q}V^{p,q}$. Next, we define the filtration (the analog of the flag decomposition, Eq.(2.19)) via $F^pV = \bigoplus_{p'>p} V^{p',q'}$ so that $\cdots \supset F^pV \supset F^{p+1}V \supset \cdots$ is a decreasing filtration on V. The differential form, Eq.(4.39), belongs to space $\Omega_m^{n+1}(\mathcal{F})$ of differential forms such that $\omega = \dfrac{p(\mathbf{z})}{q(\mathbf{z})^m}\omega_0$, where $p(\mathbf{z})$ is homogenous polynomial of degree $m \deg(q) - (n+2)$. Such differential forms have a pole of order $\leq m$. As in the standard complex analysis one can define the multidimensional analogue of residue via map $R(\omega)$: $\Omega_m^{n+1}(\mathcal{F}) \to \mathrm{H}^n(\mathcal{F}, \mathbf{C})$ given by

$$< \sigma, R(\omega) >= \frac{1}{2\pi i}\int_\sigma \omega \text{ , } \sigma \in \mathrm{H}_n(\mathcal{F}, \mathbf{C}). \tag{4.62}$$

Deligne proves that:

a) $\mathrm{H}^n(\mathcal{F}, \mathbf{C}) = \bigoplus_{\bar{c}\neq 0} \mathrm{H}^n(\mathcal{F}, \mathbf{C})_{\bar{c}}$ where

b)$\mathrm{H}^n(\mathcal{F}, \mathbf{C})_{\bar{c}} \subset F^{<c>-1}\mathrm{H}^n(\mathcal{F}, \mathbf{C})$ while the complex conjugate of $\mathrm{H}^n(\mathcal{F}, \mathbf{C})_{\bar{c}}$ is given by $\mathrm{H}^n(\mathcal{F}, \mathbf{C})_{-\bar{c}} \subset F^{n-<c>+1}\mathrm{H}^n(\mathcal{F}, \mathbf{C})$.

Thus, by construction, $\mathrm{H}^n(\mathcal{F}, \mathbf{C})_{\bar{c}}$ is of bidegree (p, q) with $p =< c > -1, q = n - p$, while its complex conjugate $\mathrm{H}^n(\mathcal{F}, \mathbf{C})_{-\bar{c}}$ is of bidegree (q, p). Obtained cohomologies are nontrivial and of Hodge-type only when $< c > \neq 1$.Finally, the procedure of extracting the residue from integral in Eq.(4.62) with ω containing pole of order m is described in the book by Hwa and Teplitz [73] and in spirit is essentially the same as in standard one variable complex analysis. Therefore, we end up again with the differential form ω, Eq.(4.39), with $< c >= 1$.However, this form will be used with phase factors $\hat{\varepsilon}_m$ instead of ξ.Physical consequences of this replacement are considered in the Appendix B.

The obtained results provide needed support for use of method of coadjoint orbits, Section 3.1., and are in accord with the Theorem 3 by Ginzburg. They provide justification for existence of physical models (discussed in the next section) associated with periods of $\mathcal{F}(N)$.

5 Designing physical model for the Veneziano-like amplitudes

5.1 Some auxiliary results

In this section we would like to demonstrate that results obtained in previous sections contain sufficient information for restoration of the underlying physical model producing the Veneziano-like amplitudes.

Let us begin with calculation of the volume of k-dimensional simplex Δ_k. To do so, it is sufficient to consider calculation of the integral of the type[28]

$$\mathrm{vol}(\Delta_k) = \int_{x_i \geq 0} dx_1 \cdots dx_{k+1} \delta(1 - x_1 - \cdots - x_{k+1}). \tag{5.1}$$

Using results from symplectic geometry [20,48] it is straightforward to show that the above integral (up to unimportant constant) is just the microcanonical partition function for the system of $k+1$ harmonic oscillators with the total energy equal to 1. To calculate such partition function it is sufficient to take into account the integral representation of the delta function. Then, the standard manipulations with integrals produce the following anticipated result:

$$\mathrm{vol}(\Delta_k) = \frac{1}{2\pi} \oint \frac{dy \exp(iy)}{(iy)^{k+1}} = \frac{1}{k!}. \tag{5.2a}$$

Clearly, for the dilated volume we would obtain instead $\mathrm{vol}(n\Delta_k) = \dfrac{n^k}{k!}$ where n is the dilatation coefficient. From this calculation we can obtain as well the volume of k-dimensional hypercube (or, more generally the convex polytope)as

$$n^k = k! vol(n\Delta_k). \tag{5.2b}$$

This result was obtained by Atiyah [43] who was inspired by earlier result by Koushnirenko [78]. It has the following interpretation. If n denotes the number of segments, say in x direction, in $\mathbf{Z}^k$ then $n+1$ is the number of lattice points associated with such segments. As in Eq.(1.4), Eq.(5.2b) produces the total number of points inside the hypercube and at its faces. This number comes as solution of the combinatorial problem of finding the total number of points with integral coordinates inside of the dilated simplex $n\Delta_k$. In Appendix B we had introduced

$$p(k,n) = \left| n\Delta_k \cap \mathbf{Z}^k \right| = \frac{(n+1)(n+2)\cdots(n+k)}{k!} \tag{5.3a}$$

as the total number of points with integral coordinates inside of the dilated simplex $n\Delta_k$. According to Stanley [79], this result provides also the number of solutions in non negative integers of the equation $x_1 + ... + x_k \leq n$. When n$\rightarrow \infty$ Gelfand et al [19] had proved (for *any* convex integral polytope Δ_k) that

$$p(k,n) = \left| n\Delta_k \cap \mathbf{Z}^k \right| = \frac{\mathrm{vol}\Delta_k}{k!} n^k + O(n^{k-1}). \tag{5.3.b}$$

If we, as Gelfand et al, put the volume of the simplex Δ_k equal to one then, $p(k,n) \simeq \dfrac{n^k}{k!}$, as before. This result is not limited to hypercubes, of course.

[28] In view of Eq.s(4.30) and (4.43) we actually need to consider integrals of the type $\int dx_1^{<c_1>} \wedge \cdots \wedge dx_{k+1}^{<c_{k+1}>} \delta(1 - x_1^N - \cdots - x_{k+1}^N)$. They indeed will be discussed later in this section. Because their computation and interpretation is almost the same as that given by Eq.(5.1) we prefer to discuss Eq.(5.1) first.

In Section 3.2. we had discussed general case appropriate for arbitrary convex polytope. We shall return to this topic later in this section.

The number $p(k,n)$ can be obtained with help of the generating function $P(k,t)$ given by

$$P(k,t) \equiv \frac{1}{(1-t)^{k+1}} = \sum_{n=0}^{\infty} p(k,n)t^n. \qquad (5.4)$$

This generating function can be interpreted as partition function of some statistical mechanical system. Indeed, in the standard bosonic string theory similar generating function

$$P(t) = \prod_{n=1}^{\infty} \frac{1}{1-t^n} = \sum_{n=0}^{\infty} p(n)t^n \qquad (5.5)$$

is used as partition function for the bosonic string[29] .

According to Apostol [80] the number $p(n)$ can be understood as follows. Every non negative number n can be represented as

$$n = (1+...+1)+(2+...+2)+...+(m+...+m)+.. = k_1+2k_2+...+mk_m+...$$

where k_1 gives number of ones in first parenthesis, k_2 gives number of twos, and so on. Hence, $p(n)$ is a partition of n into positive summands whose number is $\leq n$. Clearly, $p(k,n)$ and $p(n)$ are related to each other. The nature of their relation will be discussed below in Section 5.1.2. In the meantime, let us suppose that we can approximate the "exact" partition function $P(t)$ by $P(k,t)$ at least for large k. Then, using Eq.s (5.4) and results of the Appendix B we replace k by $\alpha(s)$ and use this expression in beta function representing Veneziano amplitude $V(s,t)$. Performing term- by- term integration we obtain Eq.(B.14) which is familiar form of the Veneziano amplitude [55,62]. Incidentally, it is straightforward to demonstrate that such integration is equivalent to taking the Laplace transform of $P(k,t)$ with respect to $t-$variable in Eq.(5.4). From general results of Regge theory it is known that such procedure (when applied to the partition function) leads to poles revealing the spectrum.

Based on the results just presented several conclusions can be drawn: a) the partition function $P(k,t)$ can be adequately used in connection with the Veneziano amplitude; b) such partition function has geometric and group-theoretic meaning. In particular, $p(k,n)$ is exactly equal to the Kostant multiplicity formula, Eq.(1.45), as it is proven in Guillemin et all [47], Chr7., and Guillemin [81], Chapter 4.[30]; c) although the Kostant multiplicity formula was extended by Kac to the infinite dimensional case (Ref.[12] and Appendix A) in the present

[29] The fact that $P(t)$ should be raised to 24th power [55] does not change much in this regard.

[30] Actually, these authors had proven the following . Let $N(\alpha)$ denote the number of integer solutions of the linear equation $\alpha = n_1\alpha_1 + ... + n_k\alpha_k$, for some prescribed $n_i \geq 0$, $\alpha_i \in \mathbf{Z}^k$ $(i = 1,...,k)$, then $N(\alpha) = k!\text{vol}(\alpha\Delta_k)$, e.g. see Eq.(5.2b) where, Δ_k is convex polytope associated with solution of the above linear equation (as explained in Section 3.2). Such obtained function $N(\alpha)$ happen to coincide with Kostant multiplicity formula, Eq.(1.45).

case we are dealing with $finite$ *dimensional* version of the Kostant multiplicity formula because it is adequate.

Just presented arguments will be considerably extended and reinforced below.

5.1.1 Additional facts from the theory of pseudo-reflection groups

In Appendix A, part c), we had listed some basic facts related to pseudo-reflection groups. At this point we would like to extend this information in connection with results obtained thus far. To this purpose we would like to use some facts from the classical paper by Shepard and Todd [82] (S-T). We shall use it along with the monograph by McMullen [83] containing the up to date developments related to S-T work.

Adopting S-T notations, the unitary group $G(N,p,n)$ is defined as follows. Let $N \geq 1, n \geq 2$, and let p be a divisor of N, i.e. $N = pq$. Let ξ be a primitive N-th root of unity. Then $G(N,p,n)$ is the group of all monomial transformations in $\mathbf{C}^n$ of the form (e.g. see Section 2.4)

$$x_i' = \xi^{\nu_i} x_{\sigma(i)},\ i = 1, ..., n, \tag{5.6}$$

where $\sigma(1), ..., \sigma(n)$ is permutation σ of $(1, , ..., n)$, i.e. $\sigma \in S_n$,and

$$\sum\nolimits_i \nu_i = 0(\operatorname{mod} N). \tag{5.7a}$$

(the Veneziano condition). In the case if $N = pq$ the above condition should be changed to

$$\sum\nolimits_i \nu_i = 0(\operatorname{mod} p) \tag{5.7b}$$

(the closed string condition). The group $G(N,p,n)$ has order(or cardinality) $|G| = qN^n n!$. This follows easily from the same arguments as were discussed in Section 2.4. In addition, the factor $n!$ is coming from permutational symmetry S_n while the factor q is counting extra possibilities coming from the decomposition of N. Alternatively, the order of the group $G(N,p,n)$ can be reobtained by considering the set of 2-fold reflections given by

$$x_i' = \xi^{\nu} x_j,\quad x_j' = \xi^{-\nu} x_i,\quad x_k' = x_k\ , k \neq i, j. \tag{5.8}$$

Such set generates a normal subgroup of order $N^n n!$. The other reflections, if any, are of the form

$$x_j' = \xi^{\frac{\nu N}{r}} x_j,\ x_i' = x_i \text{ for } i \neq j, \tag{5.9}$$

where $j = 1, ..., n$, $(\nu, N/r) = 1$and $r \mid q$ if $q > 1$. The following theorem can be found in McMullen's book [83], page 292.

Theorem 4. *If $n \geq 2$, then up to conjugacy within the group of all unitary transformations, the only* ***finite*** *irreducible unitary reflection groups in $\mathbf{C}^n$ which are imprimitive are the groups $G(m,p,n)$ with $m \geq 2, p \mid m$ and $(m,p,n) \neq (2,2,2)$.*

Definition 16. A group G of unitary transformations of $\mathbf{C}^n$ is called imprimitive if $\mathbf{C}^n$ is the direct sum $\mathbf{C}^n = E_1 \oplus \cdots \oplus E_k$ of non-trivial proper linear subspaces $E_1, ..., E_k$ such that the family$\{E_1, ..., E_k\}$ is invariant under G.

Remark 10. Clearly, the flag decomposition discussed in Section 2.4. fits this criteria.

Remark 11. The above Theorem 4 provides justification for developments presented in this paper thus far and to those which follow. Clearly, it is just a restatement of the results presented earlier in Section 2.4.

For our purposes, it is sufficient to consider only the case $p = 1$.The group $G(N, 1, n)$, traditionally denoted as γ_n^N , is the group of symmetries of the complex $n-$ cube. Actually, it is the same as that for real n-cube [82,84] which is inflated by the factor of N. Thus, we have reobtained the result, Eq.(1.4), of Section 1.1. (given that the length N contains $N+1$ points (including the origin)) The cubic symmetry is also discussed in the Appendix A. In spite of these facts, in order to utilize them efficiently we need to reobtain the same results from yet another perspective. To this purpose, following S-T we introduce an auxiliary function $g_r(m, p, n)$ describing the number of operations of $G(m, p, n)$ which leave fixed every point of subspace of dimensionality $n-r, r = 0, 1, ..., n$. By definition, $g_0(m, p, n) = 1$ and, moreover, let

$$G(m, p, n; t) \equiv \sum_{r=0}^{n} g_r(m, p, n) t^r. \tag{5.10}$$

On one hand, the above equation serves as a definition of the generating function $G(m, p, n; t)$, on another- in view of the results of Section 1, we can interpret the r.h.s. as the Weyl character formula. Shepard and Todd calculate $G(m, p, n; t)$ explicitly. Their derivation is less physically adaptable however than that of Solomon [8]. Hence, we would like to discuss Solomon's results now.

5.1.2 Selected exercises from Burbaki (end)

Section 4.3. contains some solutions to the problem set #3 given at the end of Chapter 5 (paragraph 5) of Bourbaki [7]. In view of their physical significance,we would like to present the rest of the solutions now. The problem set # 3 deals with results obtained by Solomon [8]. In this subsection we would like to reobtain his results in physically more illuminating way. In doing so we shall freely use notations and results of Section 4.3. In particular, we need to use the ring $S(V)^G$ of symmetric invariants composed of algebraically independent polynomial forms $P_1, ..., P_l$ made of symmetric combinations of $t_1, ...t_l$ raised to some powers d_i , $i = 1, ..., l$, different for different reflection groups [84]. The ring of invariants is graded and it admits decomposition (which is actually always finite) :$S(V)^G = \bigoplus_{j=0}^{\infty} S_j(V)^G$. Let $\dim_K V_j^G$ be the dimension of the graded invariant subspace $S_j(V)^G$ defined over the field K. Then, the Poincare$'$ polynomial $P(S(V)^G, t)$ is defined by

$$P(S(V)^G, t) = \sum_{i=0}^{\infty} (\dim_K V_j^G) t^i. \tag{5.11}$$

The Poincare' polynomial possesses the splitting property [84](the most useful in K-theory) of the following nature. If the total vector space M is made as a product $V \otimes_K V'$ of vector spaces V and V' then, the Poincare' polynomial is given by

$$P(V \otimes_K V', t) = P(V, t)P(V', t). \tag{5.12}$$

This splitting property is of topological nature and is extremely useful in actual calculations. In particular, consider the polynomial ring F$[x]$ made of monomials of degree d which can be identified with the graded vector space V. Then, the Poincare polynomial for such space is given by

$$P(V, t) = 1 + t^d + t^{2d} + ... = \frac{1}{1 - t^d}. \tag{5.13}$$

Consider now the multivariable polynomial ring F$[x_1, ..., x_n]$ made of monomials of respective degrees d_i . Then, using the splitting property we obtain at once

$$V^T = \mathrm{F}[x_1] \otimes_K \mathrm{F}[x_2] \otimes_K \mathrm{F}[x_3] \otimes_K \cdots \otimes_K \mathrm{F}[x_n]$$

and, of course,

$$P(V^T, t) = \frac{1}{1 - t^{d_1}} \cdots \frac{1}{1 - t^{d_n}}. \tag{5.14}$$

In particular, if all d_i in Eq.(5.14) are equal to one, which is adequate for $S(V)$,e.g. read Ref.[84] , page 171, then we reobtain back Eq.(5.4) (with n in Eq.(5.14) being replaced by $k + 1$). At the same time, in the case of cubic symmetry B_{n+1}(useful for the purposes of this work), S-T find for the group $G(N, 1, n)$ the exponents $d_i = Ni$ with $i = 1, ..., n$ [82-84]. In the typical case of real space and cubic symmetry we have $N = 2$ (e.g. see Eq.(A.6) of Appendix A) so that these exponents $d_i$coincide with those listed in the book by Humphreys[85], page 59, as required. If we replace t by $t^2 = x$ (or, more generally, $t^N = x$) in the associated Poincare' polynomial, then we obtain,

$$P(V^T, x) = \prod_{i=1}^{k+1} \frac{1}{1 - x^i} = \sum_{n=0}^{\infty} \hat{p}(k, n)x^n. \tag{5.15}$$

In the limit $k \to \infty$ this result coincides with Eq.(5.5). According to the Theorem 4 cited above the arbitrary large n still leads to **finite** unitary reflection group. It is tempting to choose the Poincare' polynomial, Eq.(5.15), as partition function for "new" bosonic string. This is not the case however. The new string partition function is not given by Eq.(5.15). It does not correspond to the truncated bosonic string model and, in fact, it is not even bosonic! This is so because of the results obtained by Solomon [8] in 1963 to be discussed now.

To make our presentation self contained and focused on physics at the same time we need to discuss few auxiliary facts from the theory of invariants of the Weyl-Coxeter reflection groups. Let $G \subset GL(V)$ be one of such groups. Denote $|G| = \prod_i d_i$ and introduce the averaging operator $Av : V \to V$ via

$$Av(x) = \frac{1}{|G|} \sum_{\varphi \in G} \varphi \circ x. \tag{5.16}$$

By definition, x is the group invariant, $x \in V^G$, if $Av(x) = x$. In particular,

$$\dim_K V_j^G = \frac{1}{|G|} \sum_{\varphi \in G} tr(\varphi_j). \tag{5.17}$$

By combining this result with Eq.(5.11) we obtain:

$$\begin{aligned} P(S(V)^G, t) &= \sum_{i=0}^{\infty} (\dim_K V_j^G) t^i = \sum_{i=0}^{\infty} \frac{1}{|G|} \sum_{\varphi \in G} tr(\varphi_i) t^i \\ &= \frac{1}{|G|} \sum_{\varphi \in G} [\sum_{i=0}^{\infty} tr(\varphi_i) t^i] = \frac{1}{|G|} \sum_{\varphi \in G} \frac{1}{\det(1 - \varphi t)}. \end{aligned} \tag{5.18}$$

The obtained result is known as the Molien theorem [84]. It is based on the following nontrivial identity

$$\sum_{i=0}^{\infty} tr(\varphi_i) t^i = \frac{1}{\det(1 - \varphi t)} \tag{5.19}$$

valid for the upper triangular matrices, i.e. for matrices which belong to the Borel subgroup B (Section 2). For such matrices

$$tr(\varphi_i) = \sum_{j_1 + j_2 + \ldots + j_n = i} \lambda_1^{j_1} \cdots \lambda_n^{j_n} \tag{5.20}$$

where the Borel-type matrix φ of dimension n has $\lambda_1, ..., \lambda_n$ on its diagonal. In view of Eq.s(2.14),(2.18), and following Humphreys [85], it is useful to re interpret Eq.(5.20) as follows. We consider action of the averaging operator, Eq.(5.16), on the monomials

$$x = z_1^{j_1} \cdots z_n^{j_n}, \text{ where } j_1 + j_2 + ... + j_n = i.$$

These are the eigenvectors for φ_i with corresponding eigenvalues $\lambda_1^{j_1} \cdots \lambda_n^{j_n}$.The sum of these eigenvalues is the trace of the linear operator $Av(x)$ on $S_i(V)^G$. But, according to Eq.(5.17), this is just the dimension of space $S_i(V)^G$. This dimension has the following meaning. If, as we had argued after Eq.(2.22), the eigenvalues in Eq.(5.20) are made of $i - th$ roots of unity then, by combining Eq.s (2.16), (2.20),(2.22), Appendix A, part c), and Eq.(5.16) we arrive at the Veneziano condition

$$\sum_k m_k j_k = i \tag{5.21a}$$

again. Since $m_i = d_i - 1 \bmod i$, the above equation is equivalent to

$$\sum_k d_k j_k = 0 \bmod i \tag{5.21b}$$

with the exponents d_i defined earlier. In particular, if $\sum_k d_k j_k = i$ then, using this equation along with Eq.s(5.19),(5.20), we arrive at the following physically suggestive result:

$$\begin{aligned}\sum_{i=0}^{\infty} tr(\varphi_i)t^i &= \sum_{i=0}^{\infty}[\sum_{j_1d_1+j_2d_2+...+j_nd_n=i} \lambda_1^{j_1d_1}\cdots\lambda_n^{j_nd_n}]t^i \\ &= \sum_{j_1=0}^{\infty} t^{j_1d_1}\sum_{j_2=0}^{\infty} t^{j_2d_2}\cdots\sum_{j_n=0}^{\infty} t^{j_nd_n} \\ &= \prod_{i=1}^{n}\frac{1}{1-t^{d_i}} \end{aligned} \tag{5.22a}$$

to be compared with earlier obtained Eq.(5.14). Alternatively,

$$\sum_{i=0}^{\infty} tr(\varphi_i)t^i = \sum_{j_1=0}^{\infty}\lambda_1^{j_1}t^{j_1}\cdots\sum_{j_n=0}^{\infty}\lambda_n^{j_n}t^{j_n} = \frac{1}{\det(1-\varphi t)}. \tag{5.23}$$

We have gone through all details in order to demonstrate the bosonic nature of the obtained result: by replacing t with $\exp(-\varepsilon)$ with $0 \leq \varepsilon \leq \infty$ and associating numbers j_i with the Bose statistic occupation numbers we have obtained the partition function for the set of n independent harmonic oscillators (up to zero point energy). By combining these results with Eq.(5.18) we obtain yet another physically suggestive (to those who prefer to use the bosonic path integrals) result:

$$\frac{1}{|G|}\sum_{\varphi\in G}\frac{1}{\det(1-\varphi t)} = \prod_{i=1}^{n}\frac{1}{1-t^{d_i}}. \tag{5.24}$$

The results just obtained are auxiliary however. Our main objective is to obtain the explicit form of $G(m,p,n;t)$ defined in Eq.(5.10) and to explain its physical meaning. To this purpose, let us recall that according to the Theorem 2 by Solomon the differential form $\omega^{(p)}$, Eq.(4.26), belongs to the set of G-invariants of the product $S(V)\otimes E(V)$. The splitting property, Eq.(5.12), of the Poincare' polynomials requires some minor changes for the present case. In particular, if by analogy with $S(V)^G$ decomposition we would write $(S(V)\otimes E(V))^G = \bigoplus_{i,j} S_i(V)^G\otimes E_j(V)^G$, then, the associated Poincare' polynomial is given by

$$P((S(V)\otimes E(V))^G; x, y) = \sum_{i,j\geq 0}(\dim_K S_i^G\otimes E_j^G)x^iy^j. \tag{5.25}$$

By analogy with Eq.(5.17), following Solomon [8], we introduce

$$\dim_K S_i^G\otimes E_j^G = \frac{1}{|G|}\sum_{\varphi\in G} tr(\varphi_i)tr(\varphi_j). \tag{5.26}$$

In order to use this result we need to take into account Eq.(1.18). That is we need to take into account that

$$\sum_{j=0}^{n} tr(\varphi_j)y^j = \det(1+\varphi y) \tag{5.27}$$

to be contrasted with Eq.(5.19). To prove that this is the case it is sufficient to recall that for fermions the occupation numbers j_i are just 0 and 1. Hence,

$$\begin{aligned}\sum_{j=0}^{n} tr(\varphi_j)y^j &= \sum_{j=0}^{n}[\sum_{j_1+j_2+\ldots+j_n=j} \lambda_1^{j_1}\cdots\lambda_n^{j_n}]y^j \\ &= \prod_{i=1}^{n}\sum_{j_i=0}^{1}\lambda_i^{j_i}y^{j_i} = \prod_{i=1}^{n}(1+\lambda_i y), \end{aligned} \tag{5.28}$$

to be compared with Eq.(5.22b).Using Eq.(5.26) in (5.25) and taking into account the rest of the results, in accord with Bourbaki [7] the following expression for the Poincare$'$ polynomial is obtained :

$$P((S(V)\otimes E(V))^G;x,y) = \frac{1}{|G|}\sum_{\varphi\in G}\frac{\det(1+\varphi y)}{\det(1-\varphi x)} = \prod_{i=1}^{n}\frac{(1+yx^{d_i-1})}{(1-x^{d_i})}. \tag{5.29a}$$

To check its correctness we can: a) put $y=0$ thus obtaining back Eq.(5.23) or b) put $y=-x$ thus obtaining identity $1=1$.

Remark 12. Comparison of Eq.(5.29a) with Ruelle zeta function, Eq.(1.24), suggests the interpretation of this result from the point of view of evolution of dynamical systems. Additional details concerning such interpretation can be found in our earlier work, Ref. [2].

The result, Eq.(5.29a), can now be used for several tasks. First, for completeness of presentation we would like to recover the major S-T result:

$$G(m,p,n;t) = \prod_{i=1}^{n}(m_i t+1) \tag{5.30}$$

used heavily in theory of hyperplane arrangements [60,86]. For $t=1$ above equation produces: $G(m,p,n;t=1) = |G| = \prod_{i=1}^{n} d_i$ as required. Moreover, in view of Eq.(5.10), it allows us to recover $g_r(m,p,n)$. Second, after this is done, we need to discuss its physical meaning.

To recover the S-T results let us rewrite Eq.(5.29) as follows

$$\sum_{\varphi\in G}\frac{\det(1+\varphi y)}{\det(1-\varphi x)} = |G|\prod_{i=1}^{n}\frac{(1+yx^{d_i-1})}{(1-x^{d_i})} \tag{5.29b}$$

and let us treat the right (R) and the left (L) hand sides separately. Following Bourbaki [7] we put $y=-1+t(1-x)$. Substitution of this result to R produces

at once

$$R\,|_{x=1}=\prod_{i=1}^{n}(d_i-1+t)=\prod_{i=1}^{n}(m_i+t) \tag{5.31}$$

To do the same for L requires us to keep in mind that $\det AB=\det A\det B$ and, hence, $\det AA^{-1}=1$ leads to $\det A^{-1}=1/\det A$ .Therefore, after few steps we arrive at

$$L\,|_{x=1}=\sum_{r=1}^{n}h_r t^r. \tag{5.32}$$

Equating L with R, replacing t by $1/T$ and relabeling $1/T$ again by t and h_l by $\tilde{h}_l=g_r(m,p,n)$ we obtain the S-T result, Eq.(5.10). To obtain physically useful result we have to take into account that for the cubic symmetry we had already : $d_i=iN$. Let therefore $y=-x^{Nq+1}$ in Eq.(5.29a) so that we obtain

$$P((S(V)\otimes E(V))^G;z)=\prod_{i=1}^{n}\frac{1-z^{q+i}}{1-z^i} \tag{5.33}$$

Remark 13. The result almost identical to our Eq.(5.33) was obtained some time ago in the paper by Lerche et al.[87], page 444. To obtain their result, it is sufficient to replace z^{q+i} by z^{q-i}. Clearly, such substitution is not permissible in our case because for cubic symmetry in the limit $z=x^N=1$ we obtain

$$P((S(V)\otimes E(V))^G;z=1)=\frac{(q+1)(q+2)\cdots(q+n)}{n!} \tag{5.34}$$

in accord with Eq.(5.3a)[31].

Obtained result suggest us to choose the Poincare' polynomial, Eq.(5.29a), as partition function associated with the Veneziano amplitude. From our detailed derivation of Eq.(5.29a) it follows that: a) the underlying physical model should be supersymmetric; b) it should be finite dimensional; c) in view of the results of Section 1 the Poincare' polynomial, Eq.(5.33), could be identified with the Weyl character formula; d) accordingly, following Guillemin et al [20,48,81], the result Eq.(5.34), can be identified with the Kostant multiplicity formula.

Being armed with such results, many additional details are provided in the next subsection

5.2 The multiparticle Veneziano-like amplitudes in the light of Shepard-Todd results

Earlier, when we discussed the volume integral, Eq.(5.1), we noticed that its calculation closely resembles that of the Dirichlet-Veneziano integral given below

$$I=\int_{x_i\geq 0}dx_1^{<c_1>}\wedge\cdots\wedge dx_{k+1}^{<c_{k+1}>}\delta(1-x_1^N-\cdots-x_{k+1}^N). \tag{5.35}$$

[31]The correctness of this result can be checked straightforwardly. Indeed, if taking into account the Theorem 2 by Solomon, we consider products of the type $dx_1^{p_1}\cdots dx_n^{p_n}$ with $p_1+...+p_n=q$, then Eq.(5.34) gives the number of such products.

We would like to demonstrate now that this is indeed the case. In view of the results of Appendix B let $< c_i >= p_i$ and let N divide all $p_i^{\prime}s$ so that $n_i = \frac{N}{p_i}$. Let $x^{p_i} = z_i$ then, using the integral representation of delta function we obtain (up to a constant as before)

$$I = \frac{1}{2\pi} \oint dy [\prod_{i=1}^{k+1} \left(\frac{1}{iyn_i}\right)^{\frac{1}{n_i}} \Gamma(\frac{1}{n_i})] \exp(iy) \doteq \frac{\Gamma(\frac{p_1}{N}) \cdots \Gamma(\frac{p_{k+1}}{N})}{\Gamma(\sum_i \frac{p_i}{N})} \tag{5.36}$$

in agreement with Eq.(4.57). In order to make connection with the results earlier obtained we need to reconsider the calculation we just made. To this purpose, using again Appendix B we remove the bracket sign $< ... >$ from $c_i^{\prime}s$ and assume that p_i/N can have any rational value which we denote by x_i . With such notations our integral I acquires the form of the Dirichlet integral, Eq.(4.21). In this integral let $t = u_1 + ... + u_k$. This allows us to use already familiar expansion, Eq.(5.4), where now t^n on the r.h.s. of Eq.(5.4) we have to replace by the following identity

$$t^n = (u_1 + ... + u_k)^n = \sum_{n=(n_1,...,n_k)} \frac{n!}{n_1! n_2! ... n_k!} u_1^{n_1} \cdots u_k^{n_k} \tag{5.37}$$

with restriction $n = n_1 + ... + n_k$.This type of identity was used earlier in our work on Kontsevich-Witten model [32]. Moreover, from the same paper we obtain the alternative and very useful form of the above expansion

$$(u_1 + ... + u_k)^n = \sum_{\lambda \vdash k} f^{\lambda} S_{\lambda}(u_1, ..., u_k) \tag{5.38}$$

where the Schur polynomial S_{λ} is defined by

$$S_{\lambda}(u_1, ..., u_k) = \sum_{n=(n_1,...,n_k)} K_{\lambda,n} u_1^{n_1} \cdots u_k^{n_k} \tag{5.39}$$

with coefficients $K_{\lambda,n}$ known as Kostka numbers [89], f^{λ} being the number of standard Young tableaux of shape λ and the notation $\lambda \vdash k$ meaning that λ is partition of k. Through such connection with Schur polynomials one can develop connections with Kadomtsev-Petviashvili (KP) hierarchy of nonlinear exactly integrable systems on one hand and with the theory of Schubert varieties on another [32]. Due to the length of this paper, detailed analysis of such connections is left for future work. Nevertheless, we shall encounter below additional examples of striking similarities between the results of this paper and those of the Kontsevich-Witten model.

Using of either Eq.(5.37) or (5.38) in Eq.(5.4) and substitution of the r.h.s, of Eq.(5.4) into the Dirichlet integral, Eq.(4.21), produces (after effectively performing the multiple Laplace transform) the following part of the multiparticle Veneziano amplitude

$$A(1, ...k) = \frac{\Gamma_{n_1...n_k}(\alpha(s_{k+1}))}{(\alpha(s_1) - n_1) \cdots (\alpha(s_k) - n_k)}. \tag{5.40}$$

This result for the multiparticle Veneziano amplitude clearly should be symmetrized in order to obtain the full multiparticle Veneziano amplitude. This is accord with results by Veneziano for the 4-particle case. It also should be corrected by the appropriate inclusions of the phase factors as it was discussed already in connection with Eq.(4.58) and in Appendix B. Since the result, Eq.(5.40), was obtained with help of Eq.(5.4) all earlier arguments developed for the four-particle amplitude remain unchanged in the multiparticle case. However, this time we can do much better job with help of the result by Solomon, Eq.(5.29a).

To this purpose we would like to demonstrate that using Eq.(5.29a) we can obtain the generating (or partition) function $\mathbf{\Theta}(t,y,k)$ for the Veneziano-like amplitudes. Following Hirzebruch and Zagier [90], page 190, we replace Eq.(5.29a) by the related expression, $\tilde{\mathbf{\Theta}}(t,y,k)$

$$\tilde{\mathbf{\Theta}}(t,y,k)=\prod_{i=1}^{k}\frac{1+yt^{d_i}d\mathrm{u}_i^{d_i}}{1-t^{d_i}\mathrm{u}_i^{d_i}} \tag{5.41}$$

containing the differential form $d\mathrm{u}_i^{d_i}\dot{=}\mathrm{u}_i^{d_i-1}d\mathrm{u}_i$. If we assign the weight d_i to the power $\mathrm{u}_i^{d_i}$ then the auxiliary parameter t will keep track of such weight. In particular, should we use $\mathrm{u}_i^{d_i-1}$ instead of $d\mathrm{u}_i^{d_i}$ we would need to replace t^{d_i} by t^{d_i-1} in the numerator. For $\mathrm{u}_i=1\ \forall i$, we would reobtain back Eq.(5.29a). It is straightforward to demonstrate that the partition function $\tilde{\mathbf{\Theta}}(t,y,k)$ reproduces the entire Solomon's algebra of differential forms $\omega^{(p)}$, Eq.(4.26). Indeed, by expanding both the numerator and the denominator and taking into account Eq.(5.25) let us consider the representative term of such expansion proportional to y^pt^l where $0\leq p\leq k$. Evidently, the numerator of Eq.(5.41) will supply $t^{d_{i_1}}\cdots t^{d_{i_p}}d\mathrm{u}_{i_1}^{d_{i_1}}\cdots d\mathrm{u}_{i_p}^{d_{i_p}}$ contribution while the denominator will supply $t^{e_1d_{i_1}}\cdots t^{e_pd_{i_p}}\mathrm{u}_{i_1}^{e_1d_{i_1}}\cdot\cdot\cdot\mathrm{u}_{i_p}^{e_pd_{i_p}}$ contribution to such term with numbers e_i being nonnegative integers. Multiplication of these two factors should be done under the condition that such term (as well as any other term in the expansion of $\tilde{\mathbf{\Theta}}(t,y,k)$) is to be $G-$invariant and, in our case,of weight l. This implies that the sum $e_1d_1+...+e_pd_p=l-d_{i_1}-...-d_{i_p}$. Finally, antisymmetrization of the product $d\mathrm{u}_{i_1}^{d_{i_1}}\cdots d\mathrm{u}_{i_p}^{d_{i_p}}$ would lead to $\frac{1}{p!}d\mathrm{u}_{i_1}^{d_{i_1}}\wedge\cdots\wedge d\mathrm{u}_{i_p}^{d_{i_p}}$. Fortunately, in Eq.(4.26) this is done already so that the product $d\mathrm{u}_{i_1}^{d_{i_1}}\cdots d\mathrm{u}_{i_p}^{d_{i_p}}$ is sufficient.

To make physical sense out of the results just presented we need to change the rules slightly. These are summarized in the following partition function

$$\mathbf{\Theta}(t,y,k)=\frac{1}{\alpha_1\cdots\alpha_k}\prod_{i=1}^{k}\pi(k;j)\frac{1+y(k)d\mathrm{u}_i^{\alpha_i}}{1-t^{d_i}\mathrm{u}_j^{d_i}} \tag{5.42}$$

where the notation $y(k)$ symbolizes the fact that we are interested only in terms proportional to $k-th$ power of y. Also,we have replaced the group exponents d_i in numerator by arbitrary nonzero integers α_i (in view of Eq.(4.21) and Appendix B) and introduced the symmetrization operator $\pi(k;j)$ which is

essentially of same nature as symmetrization in Eq.(5.37). By expanding both numerator and denominator and collecting terms proportional to $y^k t^n$ we obtain the following expression for these terms

$$t^{e_1 d_{i_1}} \cdots t^{e_k d_{i_k}} \mathrm{u}_{i_1}^{e_1 d_{i_1}} \cdots \mathrm{u}_{i_k}^{e_k d_{i_k}} \{\mathrm{u}_1^{\alpha_1} \cdots \mathrm{u}_k^{\alpha_k}\} d\mathrm{u}_1 \cdots d\mathrm{u}_k.$$

As before, we have to require $e_1 d_1 + ... + e_k d_k = n$. We had encountered and discussed this expression earlier (after Eq.(5.5)) since the exponents d_i are given by i with $i = 1, ..., k$.Once we have a combination of nonnegative integers $n_1 + ... + n_k = n$ it can be represented as $d_1 e_1 + ... + d_k e_k = n$ so that the number of ways to compose n out of n_i's is given by Eq.(5.3a).This factor should be superimposed with the symmetrizing factor of the type given in Eq.(5.37) before the above expression is ready for integration. Integrating each u_i variable within limits [0,1] we reobtain back the portion of the Veneziano amplitude given by Eq.(5.40).

Remark 14. a) The factor $\frac{1}{\alpha_1 \cdots \alpha_k}$ in front of Eq.(5.42) could be avoided should we replace $d\mathrm{u}_i^{\alpha_i}$ in the numerator of Eq.(5.42) by $\mathrm{u}_i^{\alpha_i - 1}$ and use the factors $d\mathrm{u}_i$ together with integration operation. b) To represent just obtained canonical formalism results in the grand canonical form causes no additional problems and, hence, for the sake of space is left for the readers. c) Obtained canonical expression does not contain the essential phase factors. These should be reinstated in order the above expression can be actually used.

Based on these results, the Veneziano-Shepard-Todd-Solomon canonical partition function is given by

$$\mathbf{\Theta}(t, y, k) = \prod_{i=1}^{k} \pi(k; j) \frac{1 + y(k)\mathrm{u}_i^{\alpha_i - 1}}{1 - t^{d_i} \mathrm{u}_j^{d_i}}. \tag{5.43}$$

5.3 The moment map, the Duistermaat-Heckman formula and the Khovanskii-Pukhlikov correspondence

In Section 3.2 we had introduced and discussed the moment map. Based on the results obtained thus far we are ready to use this map now. To this purpose it is sufficient to replace the integral I in Eq.(5.35) by much simpler integral

$$I(E) = \int_{x_i \geq 0} dx_1^{c_1} \cdots dx_{k+1}^{c_{k+1}} \delta(E - x_1 - \cdots - x_{k+1}) \tag{5.44}$$

where the parameter E is introduced for further convenience. The trivial case, when $1 = c_1 = ... = c_{k+1}$, was discussed already, e.g. see Eq.s(5.1) and (5.2), so that for this case: $I(E) = E^k/k!$ .The general case can be treated similarly and it will be discussed in the next subsection. In the meantime, for this "trivial" case the partition function Ξ, Eq.(1.42), can be easily obtained as

$$\Xi(\beta) = \int_0^\infty dE I(E) \exp(-\beta E) = \beta^{-(k+1)}. \tag{5.45}$$

By combining Eq.s(5.44) and (5.45) this result can be rewritten in the alternative form

$$\Xi(\beta) = \int dx_1 \cdots dx_{k+1} \exp(-\beta(x_1 + ... + x_{k+1})). \tag{5.46}$$

Taking into account that $x_i \geq 0\ \forall i$ we reobtain $\Xi(\beta) = \beta^{-(k+1)}$ as required. Such seemingly trivial results can be rewritten in the form compatible with the Duistermaat-Heckman (D-H) formula [20,48,81]. In order to use and to understand this formula several definitions need to be introduced.

In particular, as it was mentioned already in Ref. [2], the Fermat variety $\mathcal{F}(N)$ defined by Eq.(4.36) is a special case of the Brieskorn-Pham (B-P) variety V_{B-P} defined by

$$V_{B-P}(f) = \{\mathbf{z} \in \mathbf{CP}^{n+1} \mid f(\mathbf{z}) = 0\} \tag{5.47}$$

where $f(\mathbf{z}) = z_0^{a_0} + ... + z_{n+1}^{a_{n+1}}$ with $a_0, ..., a_{n+1}$ being a $n+2$ tuple of positive integers greater or equal than 2.Clearly, $\mathcal{F}(N)$ is a special case for which all a_i's are the same and equal to N. By analogy with Eq.(2.14) consider now the torus action (the monodromy) map $h : V_{B-P}(f) \to V_{B-P}(f)$ defined by

$$h_t(\mathbf{z}) \equiv h_t(z_0, ..., z_{k+1}) = \{\exp(it/a_0)z_0, ..., \exp(it/a_{k+1})z_{k+1}\}. \tag{5.48}$$

Let now $I : 0 \leq t \leq 2\pi$ so that the mapping torus T_h fiber bundle can be constructed according to the following recipe:

$$T_h = \frac{V_{B-P}(f) \times I}{i}, \tag{5.49}$$

where the identification map i is defined by the rule

$$i : (\mathbf{z}, 0) = (h_{2\pi}(\mathbf{z}), 2\pi). \tag{5.50}$$

Thus constructed fiber bundle has topological and dynamical meaning (by the way, this also provides yet another reason for inclusion of the Ruelle transfer operator Eq.(1.13) in Section 1) discussed in earlier work [2]. For the sake of space we refer our readers to this work for additional details.

Since such constructed fiber bundle "lives" in the complex projective space it also has a meaning of a symplectic manifold M. Using the same methods as in Sections 3.2. and 4.4.2. we construct a moment map $\mathcal{H}[\mathbf{z}] : M \to P$ which reduces M to a polytope P (in our case to a simplex Δ which is also a polytope). This can be easily understood if in the period integral $I = \oint \omega$ with ω given by Eq.(4.39) we (for the sake of argument) choose $< c >= 1$ in the denominator then, use the identity $1/a = \int_0^\infty dt \exp(-at)$ to bring the denominator into exponent, and then, perform the same deformation retract operations which resulted in Eq.(4.54). On such deformation retract the indentification map i still works. In general, for each fiber $f_i(\mathbf{z}) = e_i \in P$ (so that $\sum_i m_i e_i = \mathcal{E}$, e.g. see Eq.(3.10)) we introduce the reduced phase space known as the Mardsen-Weinstein symplectic quotient $M_{red}(\mathcal{E}) = \{f_i(\mathbf{z}) = e_i \mid \mathbf{z} \to h_{2\pi}(\mathbf{z})\}$ [51]. Clearly, since the vertices of P are fixed points of M, $M_{red}(\mathcal{E})$ contains

singularities associated with these fixed points. Take now the moment map, Eq.(3.10), and consider the integral

$$I[\mathbf{m}] = \int\limits_{M_{red}} \exp(- < \mathbf{m} \cdot \mathbf{f}(\mathbf{x}) >) d\mathbf{x} \tag{5.51}$$

where $d\mathbf{x}$ is the symplectic measure (in case of Eq.(5.44) it can be reduced to $dx_1...dx_{k+1}$, e.g. see the next subsection) and $< \mathbf{m} \cdot \mathbf{f}(\mathbf{x}) >= \sum_{i=1}^{k+1} m_i f_i(\mathbf{z})$ (which in the case of Eq.(5.46) is just $\sum_{i=1}^{k+1} x_i$). The related integral, known as the D-H formula, can be written as follows [20,51]:

$$I[\boldsymbol{\beta}; \boldsymbol{\xi}] = \int\limits_{P} \exp(-i\beta < \boldsymbol{\xi} \cdot \mathcal{E} >) h(\mathcal{E}) d\mathcal{E} \ . \tag{5.52}$$

It can be thought as the Fourier transform of the D-H function $h(\mathcal{E})$ which is just the characteristic function of P(the push forward of the symplectic measure), i.e. it is 1 for points $\mathcal{E} \in P$ and 0 otherwise. In agreement with the result by Atiyah [43],

$$I[\boldsymbol{\beta} \rightarrow 0] = vol M_{red}(P) \equiv vol P, \tag{5.53}$$

where in our case $volP$=$p(k,n)$ (with $n = \mathcal{E}$) was defined in Eq.(5.3a)[32]. This is so because $M_{red}(\mathcal{E})$ is made of collection of single points as discussed in Ref. [48], page 71, and Section 2.4. In particular, in Subsection 2.4. it was argued that if $M_{red}(\mathcal{E})$ can be thought of as the projective toric variety, then such variety is made of points. For non negative integer $\mathcal{E}$ each such point represents solution in nonegative integers of the equation $x_1 + ... + x_k = \mathcal{E}$ as discussed earlier in this section. Moreover, in Section 3.1.it was demonstrated that $volM_{red}(\mathcal{E}) = \chi$ where χ is the Euler characteristic of such type of manifolds[33]. This information is to be used in the next subsection. In the meantime, D-H had also demonstrated that

$$I[\beta; \boldsymbol{\xi}] = \sum_{p} \frac{\exp(-i\beta < \boldsymbol{\xi} \cdot \mathbf{f}(p) >)}{(i\beta)^k \prod\limits_{i}^{k} < \mathbf{m}_p \cdot \boldsymbol{\xi} >} = \int\limits_{M_{red}} \exp(-i\beta < \boldsymbol{\xi} \cdot \mathbf{f}(\mathbf{x}) >) d\mathbf{x} \tag{5.54}$$

with vector $\mathbf{m}_p$ representing the set of weights associated with p-th solution of Eq.(3.9) (with $|z_i|^2$ being replaced by x_i) while ξ is any vector consistent with definition of the rational polyhedral cone (Section 2.1). By expanding both

[32] Comparing this "quantum"result against "classical", Eq.(5.2), we notice that the "semi-classical approximation": n$\rightarrow \infty$ of quantum result produces the classical result as in standard quantum mechanics.

[33] Alternative derivation of this fact involving rather sophisticated methods of algebraic geometry can be found in Ref.[23].

sides of the above identity in power series in auxiliary parameter β and equating $\beta-$independent terms we obtain

$$volP = (-1)^k \sum_p \frac{<\boldsymbol{\xi} \cdot \mathbf{f}(p)>^k}{k! \prod_i^k < \mathbf{m}_p \cdot \boldsymbol{\xi} >} \tag{5.55}$$

in accord with result by Vergne[51] given without derivation. Because this equality should be independent of the choice for ξ we must require

$$k! \prod_i^k < \mathbf{m}_p \cdot \boldsymbol{\xi} >= (-1)^k < \boldsymbol{\xi} \cdot \mathbf{f}(p) >^k \tag{5.56}$$

in order to be in accord with Eq.(5.53). Instead of analyzing this equation we choose another (much shorter) route aimed at proof of Eq.(5.55) by utilizing some results from the work of Khovanskii and Pukhlikov [91]. Following these authors, we introduce auxiliary functions

$$i(x_1, ..., x_k; \xi_1, ..., \xi_k) = \frac{1}{\xi_1 ... \xi_k} \exp(\sum_{i=1}^k x_i \xi_i), \tag{5.57a}$$

$$s(x_1, ..., x_k; \xi_1, ..., \xi_k) = \frac{1}{\prod_{i=1}^k (1 - \exp(-\xi_i))} \exp(\sum_{i=1}^k x_i \xi_i). \tag{5.57.b}$$

The authors demonstrate that these functions are connected to each other with help of the Todd transform, i.e.

$$Tdi(y_1, ..., y_k; \boldsymbol{\xi})_{y_i = x_i} = s(x_1, ..., x_k; \boldsymbol{\xi}), \tag{5.58}$$

where $\boldsymbol{\xi} = (\xi_1, ..., \xi_k)$ and the Todd operator $Td(z)$ is defined by

$$Td(z) = \prod_{i=1}^k \frac{z_i}{1 - \exp(-z_i)} \mid_{z_i \to \frac{\partial}{\partial z_i}} . \tag{5.59}$$

The significance of this result comes from the following observations. First, the expression $\sum_{i=1}^k x_i \xi_i$ in Eq.(5.57) represents a convex polyhedral cone, e.g. see Eq.(2.2). As Fig.2 suggests, summation over cones forming complete fan is equivalent to summation over the vertices of the polytope. If such summation is made we arrive at the r.h.s. of the identity, Eq.(1.7), and, hence, we can use the l.h.s. of Eq.(1.7) as well. Second, since we have connected the identity, Eq.(1.7), with the Weyl character formula in Section 1, the l.h.s. of Eq.(1.7) is just the character of the Weyl reflection group, e.g. see Eq.s(1.38) and (1.39). Such character can be associated with quantum mechanical partition function, Eq.(1.40).Third, looking at the D-H result, Eq.(5.54), replacing factor $i\beta$ by

-1,readjusting $\mathbf{m}_p$ and comparing such D-H sum with Eq.(5.57a) (summed over vertices) we arrive at complete equivalence between these two expressions. Moreover, by expanding (summed over the vertices) the denominator of the r.h.s. of Eq.(5.57b) in powers of ξ in the limit of small $\xi's$ we reobtain back the D-H result, Eq.(5.54). Based on these facts, we arrive at important

Corollary 3. The Todd transform defined by Eq.(5.58) (with summation over the vertices) provides direct link between classical and quantum mechanical dynamical systems in accord with classical-quantum mechanical correspondence which follows from the method of coadjoint orbits discussed in Section 3.1.

Clearly, such correspondence holds for dynamical systems described by semisimple Lie groups (algebras) and, most likely, it can be extended to all (pseudo) reflection Weyl-Coxeter groups. In fact, for pseudo-reflection groups this fact can be considered as proven in view of Eq.s (5.10),(5.29),(5.30),(5.33) and (5.34) and results of Section 1. Moreover, this claim is supported by results of Vergne [51] who had independently obtained the central result: $volP{=}p(k,n))$ using in part the D-H formalism. No connections with Solomon-Shepard-Todd results had been mentioned in her work. For justification of the results of this paper such connections are essential.

All this can be brought to more elegant mathematical form with help of work by Atiyah and Bott [66] inspired by earlier work of Witten [92]. This is discussed in the next subsection.

5.4 From Riemann-Roch-Hirzebruch to Witten and Lefschetz via Atiyah and Bott

Let E be a vector bundle over variety X so that $ch(E)$ is the Chern character of E and $Td(X)$ is the Todd class of X , then the Hirzebruch–Grotendieck-Riemann- Roch formula the Euler characteristic $\chi(E)$ is given by [93]

$$\chi(E)=\int\limits_X ch(E)\wedge Td(X) \tag{5.60}$$

This formula is too formal for immediate applications. To connect this result with what was obtained earlier we need to use some results by Guillemin [94,95] inspired by earlier work by Pukhlikov and Khovanskii [91]. To understand the mathematical significance of the results of both references reading of earlier paper by Atiyah and Bott [66] is the most helpful.[34] Hence, we would like to make few observations related to this historic paper first. Since the rest of our paper is not written in the language of symplectic geometry needed for the current discussion, we briefly introduce few relevant notations first. In particular, for the Hamiltonian of planar harmonic oscillator discussed in Section 3.2 the standard symplectic two-form ω can be written in several equivalent ways

$$\omega=dx\wedge dy=rdr\wedge d\theta=\frac{1}{2}dr^2\wedge d\theta=\frac{i}{2}dz\wedge d\bar{z}$$

[34] In the Remark 6. we had mentioned that all details of Atiyah and Bott paper [66] are pedagogically explained in the monograph by Guillemin and Sternberg [67].

Since we are interested only in rotationally invariant observables this means that θ dependence can be dropped, i.e.[20,48,81]

$$\frac{1}{2\pi}\int\int f(\alpha r^2)rdr \wedge d\theta = \frac{1}{2}\int f(\alpha x)dx.$$

This fact was used already starting with Eq.(5.1). For collection of k oscillators the symplectic form Ω is given, as usual, by $\Omega = \sum_{i=1}^{k} dx_i \wedge dy_i = \frac{i}{2}\sum_{i=1}^{k} dz_i \wedge d\bar{z}_i$ so that its n-th power is given by $\Omega^n = \Omega \wedge \Omega \wedge \cdots \wedge \Omega = dx_1 \wedge dy_1 \wedge \cdots dx_n \wedge dy_n$. In polar coordinates the symplectic volume form is given by $(2\pi)^{-n}\Omega^n/n!$. Clearly, at this level of our presentation we can drop the factor of $(2\pi)^{-n}$. With help of these results, it is convenient to introduce the differential form

$$\exp \Omega = 1 + \Omega + \frac{1}{2!}\Omega \wedge \Omega + \frac{1}{3!}\Omega \wedge \Omega \wedge \Omega + \cdots \tag{5.61}$$

with the convention that $\int_M \hat{\omega} = 0$ if the degree of $\hat{\omega}$ differs from the dimension of M. If the manifold M does not have singularities then, according to Liouville theorem, the form Ω is closed, i.e. $d\Omega = 0$. But if M contains singularities (e.g. fixed points) this is no longer so. To fix the problem Atiyah and Bott had suggested to use the amended symplectic form

$$\Omega^* = \Omega - f \cdot u \tag{5.62}$$

with f being some function(s) (to be determined momentarily) and u being some indeterminate(s) also to be determined momentarily. The above form is (equivariantly) closed with respect to the following action of the operator d_X :

$$d_X \Omega^* = (i(X)\Omega - df) \cdot u \tag{5.63}$$

if and only if $i(X)\Omega = df$. Here $i(X)$ means the standard contraction (i.e. operation inverse to exterior differentiation). As is well known [50], the condition $i(X)\Omega = df$ is equivalent to the Hamiltonian equations and, hence, the Hamiltonian f is the moment map [5,20,48]. Consider now again the D-H integral, Eq.(5.51). In view of the results just obtained it can be rewritten as

$$I[\mathbf{m}] = \int\limits_{M_{red}} \exp \Omega^* = \int\limits_{M_{red}} \exp(\Omega - < \mathbf{m} \cdot \mathbf{f}(\mathbf{x}) >). \tag{5.64}$$

Since, when written in terms of complex variables, the form Ω represents the first Chern class, it is only natural to associate $\exp(- < \mathbf{m} \cdot \mathbf{f}(\mathbf{x}) >)$ with Chern character $ch(E)$ (with some caution). Details can be found in cited references. Clearly, the form $\mathbf{m} \cdot \mathbf{f}(\mathbf{x})$ corresponds to Atiyah and Bott's combination $f \cdot u$. It is convenient now to make formal redefinitions: $f_i \rightleftharpoons c_i$, with c_i denoting Chern class of the i-th complex line bundle[35]. Next, we temporarily replace the

[35] The splitting principle mentioned earlier allows to "disect" manifold into product of simpler manifolds, e.g. $\mathbf{C}^n = \mathbf{C} \times \mathbf{C} \cdots \mathbf{C}$,etc. each having its own first Chern class.This will be explained more carefully below.

vector **m** by indeterminate vector **-h.** After this we can use the D-H formula, Eq.(5.54), in order to rewrite it as follows

$$\int\limits_{M_{red}} \exp(\Omega + < \mathbf{h} \cdot \mathbf{c}(\mathbf{x}) >) = \sum_p \frac{\exp(< \mathbf{h} \cdot \mathbf{c}(p) >)}{\prod\limits_i^k h_i^p}. \tag{5.65}$$

Now we can apply the Khovanskii-Pukhlikov-Todd operator, Eq.(5.58), to both sides of Eq.(5.65). Clearly, the l.h.s. acquires the form

$$I[\mathbf{h}] = \sum_p \int\limits_{M_{red}} \exp(\Omega + < \mathbf{h} \cdot \mathbf{c}(p) >) \prod_{i=1}^k \frac{c_i(p)}{1 - \exp(-c_i(p))} \tag{5.66}$$

which looks essentially the same as the r.h.s of Eq.(5.60) while at the same time the r.h.s. will formally coincide with the r.h.s of Eq.(1.7) and, therefore, as it is shown in Section 1, in principle can be brought to the form coinciding with the r.h.s. of Eq.(5.33) (with n replaced by k). The Todd transform by Khovanskii and Pukhlikov allows us to obtain the Lie group characters, i.e. quantum objects, using "classical" partition functions. In particular, for $\mathbf{h} = 0$ the l.h.s. should produce the Euler characteristic χ because the r.h.s. is obviously reproducing χ. Moreover, according to Guilemin [81] , χ is equal to the dimension $Q = Q^+ - Q^-$ of the quantum Hilbert space associated with classical system described by the moment map Hamiltonian. Before discussing the meaning of subspaces $Q^+(Q^-)$ we would like for a moment to focus our attention on the mathematical meaning of the Veneziano-like amplitudes in the light of just obtained results[36]. To this purpose, following Khovanskii and Pukhlikov again, we notice that for *any* polynomial $P(z)$ we can write the identity

$$P(z_{1,\dots,}z_N) \exp(\sum_{i=1}^N p_i z_i) = P(\frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_N}) \exp(\sum_{i=1}^N p_i z_i). \tag{5.67}$$

By applying identity Eq.(5.67), to Eq.(5.65) and assuming $\mathbf{h}$=0 at the end of calculations we reobtain the result of Guillemin, Ref. [81], page 73,or Ref. [94], which is essentially the same thing as Eq.(7.17) by Atyiah and Bott. Explicitly,

$$\int\limits_{M_{red}} P(c_1, \dots, c_k) = \sum_p \frac{P(e(p)_{1,\dots,}e(p)_k)}{\prod\limits_i^k h_i^p}, \tag{5.68}$$

where the numbers $e(p)_i$ had been defined in the previous subsection. This relation had been used by Kontsevich [96] and others [97] for enumeration of rational curves on algebraic varieties. For us it is important that such relation

[36]In Section 4 we had argued that these are periods associated with differential forms living on Fermat hypersurfaces. We shall demonstrate that the arguments to be used in the remainder of this paper are in support of this fact .

can provide the intersection numbers which are averages over M_{red} of the products of the 1st Chern classes of the tautological line bundles[37]. This observation formally makes calculation of the Veneziano-like amplitudes similar in spirit to that earlier encountered in connection with the Witten-Kontsevich model [32]. From the discussion we had so far it should be clear that the l.h.s. of Eq.(5.68) is analogous to Eq.(4.54). However, below we suggest better computational alternative.

To describe this alternative, we need to make a connection with 1982 work by Witten [92]. Such connection is essential. It also was emphasized in the paper by Atiyah and Bott [66]. Fortunately, most of what we would like to say about Witten's paper is already well developed and documented [93].This allows us to squeeze our discussion to the absolute minimum emphasizing mainly features absent in standard treatments but needed for this work. In particular, we expect our readers to be familiar with the basic facts about Hodge-de Rham theory as described in 2nd edition of book by Wells [10].

We begin with the following observations. Let X be a complex Hermitian manifold and let $\mathcal{E}^{p+q}(X)$ denote the complex -valued differential forms (sections) of type (p,q) , $p+q=r$, living on X. The Hodge decomposition insures that $\mathcal{E}^r(X)$=$\sum_{p+q=r}\mathcal{E}^{p+q}(X)$. The Dolbeault operators ∂ and $\bar{\partial}$ act on $\mathcal{E}^{p+q}(X)$ according to the rule $\partial:\mathcal{E}^{p+q}(X)\rightarrow\mathcal{E}^{p+1,q}(X)$ and $\bar{\partial}:\mathcal{E}^{p+q}(X)\rightarrow\mathcal{E}^{p,q+1}(X)$, so that the exterior derivative operator is defined as $d=\partial+\bar{\partial}$. Let now φ_p,$\psi_p\in\mathcal{E}^p$. By analogy with traditional quantum mechanics we define (using Dirac's notations) the inner product

$$<\varphi_p\mid\psi_p>=\int\limits_M\varphi_p\wedge*\bar{\psi}_p \tag{5.69}$$

where the bar means the complex conjugation and the star $*$ means the usual Hodge conjugation. Use of such product is motivated by the fact that the period integrals, e.g. those for the Veneziano-like amplitudes, are expressible through such inner products [6,10]. Fortunately, such product possesses properties typical for the finite dimensional quantum mechanical Hilbert spaces. In particular,

$$<\varphi_p\mid\psi_q>=C\delta_{p,q}\text{ and }<\varphi_p\mid\varphi_p>>0, \tag{5.70}$$

where C is some positive constant. With respect to such defined scalar product it is possible to define all conjugate operators, e.g. d^*, etc. and, most importantly, the Laplacians

$$\begin{aligned}\Delta &= dd^*+d^*d,\\ \square &= \partial\partial^*+\partial^*\partial,\\ \bar{\square} &= \bar{\partial}\bar{\partial}^*+\bar{\partial}^*\bar{\partial}.\end{aligned} \tag{5.71}$$

All this was known to mathematicians before Witten's work [92]. The unexpected twist occurred when Witten suggested to extend the notion of the

[37] For the sake of space we refere our readers to Ref.[6] where these conceps are beautifully explained.

exterior derivative d. Within the de Rham picture (valid for both real and complex manifolds) let M be a compact Riemannian manifold and K be the Killing vector field which is just one of the generators of the isometry of M then, Witten had suggested to replace the exterior derivative operator d by the extended operator

$$d_s = d + si(K) \tag{5.72}$$

where the operator $i(K)$ has the same meaning as in Eq.(5.63) and s is real nonzero parameter. Witten argues that one can construct the Laplacian (the Hamiltonian in his formulation) Δ by $\Delta_s = d_s d_s^* + d_s^* d_s$. This is possible if and only if $d_s^2 = d_s^{*2} = 0$ or, since $d_s^2 = s\mathcal{L}(K)$, where $\mathcal{L}(K)$ is the Lie derivative along the field K, if the Lie derivative acting on the corresponding differential form vanishes. The details are beautifully explained in the much earlier paper by Frankel [50] discussed already in Section 3.2. What is important for us is the observation by Atiyah and Bott that if one treats the parameter s as indeterminate and identifies it with f (in Eq.(5.62)), then the derivative d_X in Eq.(5.63) is exactly Witten's d_s This observation provides the link between the D-H formalism discussed earlier in this subsection and Witten's supersymmetric quantum mechanics still to be discussed further.

In Section 4 we provided arguments explaining why the Veneziano-like amplitudes should be considered as period integrals associated with homology cycles on Fermat (hyper) surfaces. According to Wells[10], and also Ref.[6], the inner scalar product, Eq,(5.69), prior to normalization can be associated with such period integrals. Once such correspondence is established, we would like to return to our discussion of the quantum Hilbert space Q and the associated with it spaces Q^+ and Q^-. Following Ref.[48,51.81] we consider the (Dirac) operator $\acute{\partial} = \bar{\partial} + \bar{\partial}^*$ and its adjoint with respect to scalar product, Eq.(5.66), then

$$Q = \ker \acute{\partial} - co \ker \acute{\partial}^* = Q^+ - Q^- = \chi \tag{5.73}$$

Such definition was used by Vergne[51] (e.g.see discussion after Corollary 3) to reproduce Eq.(5.3a). We would like to arrive at the same result using different arguments. As a by product, we shall obtain the alternative quantum mechanical interpretation of the Veneziano-like amplitudes.

To this purpose we notice first that according to Theorem 4.7. by Wells [10] we have $\Delta = 2\square = 2\bar{\square}$ with respect to the Kähler metric on X. Next, according to Corollary 4.11. of the same reference Δ commutes with $d, d^*, \partial, \partial^*, \bar{\partial}$ and $\bar{\partial}^*$. From these facts it follows immediately that if we, in accord with Witten, choose Δ as our Hamiltonian, then the supercharges can be selected as Q$^+ = d + d^*$ and $Q^- = i\,(d - d^*)$. Evidently, this is not the only choice as Witten indicates. If the Hamiltonian H is acting in finite dimensional Hilbert space we can require axiomatically that : a) there is a vacuum state (or states) $\mid \alpha >$ such that H$\mid \alpha >= 0$ (i.e. this state is harmonic differential form) and Q$^+ \mid \alpha >=$Q$^- \mid \alpha >= 0$. This requires, of course, that [H,Q$^+$] = [H,Q$^-$] = 0. Finally, once again, following Witten we require that $(Q^+)^2 = (Q^-)^2$ =H. Then, the equivariant extension, Eq.(5.72), leads to $(Q_s^+)^2 =$ H+$2is\mathcal{L}(K)$. Fortunately,

the above supersymmetry algebra can be extended. As it was mentioned in section 3.1., there are operators acting on differential forms living on Kähler (or Hodge) manifolds whose commutators are isomorphic to $sl_2(\mathbf{C})$ Lie algebra. It was mentioned in the same section that *all* semisimple Lie algebras are made of copies of $sl_2(\mathbf{C})$. Now we can exploit these observations further using the Lefschetz isomorphism theorem whose exact formulation is given as Theorem 3.12 in Wells [10]. We are only going to use some parts of it in this work.

In particular, using notations of Ref.[10] we introduce operator L commuting with Δ and its adjoint $L^* \equiv \Lambda$. It can be shown [10], page 159, that $L^* = w * L *$ where, as before, $*$ denotes the Hodge star operator and the operator w can be formally defined through the relation $** = w$, [10], page 156. From these definitions it should be clear that L^* also commutes with Δ on the space of harmonic differential forms (in accord with page 195 of Wells). In Section 3.1. we have mentioned the Jacobson-Morozov theorem. This theorem essentially guarantees that relations given by Eq.s(3.2a-c) can be brought to form

$$[h_\alpha, e_\alpha] = 2e_\alpha \ , \ [h_\alpha, f_\alpha] = -2f_\alpha \ , \ \ [e_\alpha, f_\alpha] = h_\alpha \tag{5.74}$$

upon appropriate rescaling, e.g. see [11], page 37. Here $\alpha \in \Delta$ (Appendix A) with Δ being the root system (this notation should not cause confusion). As part of the preparation for proving of the Lefschetz isomorphism theorem, it can be shown [10] that

$$[\Lambda, L] = B \text{ and } [B, \Lambda] = 2\Lambda, \ [B, L] = -2L. \tag{5.75}$$

Comparison between the above two expressions leads to the Lie algebra endomorphism, i.e. the operators h_α, f_α and e_α act on the vector space $\{v\}$ to be described below while the operators Λ, L and B obeying the same commutation relations act on the space of differential forms. It is possible to bring Eq.s(5.74) and (5.75) to even closer correspondence. To this purpose, following Dixmier [98],Ch-r 8, we introduce the operators $h = \sum_\alpha a_\alpha h_\alpha$, $e = \sum_\alpha b_\alpha e_\alpha$, $f = \sum_\alpha c_\alpha f_\alpha$. Then, provided that the constants are subject to constraint: $b_\alpha c_\alpha = a_\alpha$, the commutation relations between the operators h, e and f are *exactly the same* as for B, Λ and L respectively. To avoid unnecessary complications we choose $a_\alpha = b_\alpha = c_\alpha = 1$. Next, following Serre [38], ch-r 4, we need to introduce the notion of the *primitive* vector (or element).This is the vector v such that hv=λv but $ev = 0$. The number λ is the weight of the module $V^\lambda = \{v \in V \mid hv{=}\lambda v\}$. If the vector space is *finite dimensional*, then $V = \sum_\lambda V^\lambda$. Moreover, only if V^λ is finite dimensional it is straightforward to prove that the primitive element does exist. The proof is based on the observation that if x is the eigenvector of h with weight λ,then ex is also the eigenvector of h with eigenvalue $\lambda - 2$, etc. Moreover, from the book by Kac [12], Chr.3, it follows that if λ is the weight of V then $\lambda - <\lambda, \alpha_i^\vee> \alpha_i$ is also the weight with the same multiplicity. Since according to Eq.(A.2) $<\lambda, \alpha_i^\vee> \in \mathbf{Z}$ **,** Kac introduces another module: $U = \sum_{k \in \mathbf{Z}} V^{\lambda + k\alpha_i}$. Such module is finite for finite reflection groups and is infinite for the affine reflection groups. We would like to argue that for our purposes it is sufficient to use only the finite reflection (or

pseudo-reflection) groups. This is so in view of the Theorem 2 by Solomon valid for finite reflection groups. It should be clear, however, from reading of book by Kac that the infinite dimensional version of the module U leads straightforwardly to all known string-theoretic results. In the case of CFT this is essential, as we had argued earlier in Sections 2.4. and 4.4.3., but for calculation of the Veneziano-like amplitudes this is not essential since by accepting such option we loose our connections with the Lefschetz isomorphism theorem (relying heavily on the existence of primitive elements) and, therefore, with the Hodge theory on which arguments of our presentation are based. Hence, below we choose to work with finite dimensional option only and in Section 6 we briefly discuss possible modifications which could be caused by variations of complex Hodge structure (i.e. motions in moduli space).

Let v be a primitive element of weight λ then, following Serre, we let $v_n = \frac{1}{n!}e^n v$ for $n \geq 0$ and $v_{-1} = 0$, so that

$$\begin{aligned} hv_n &= (\lambda - 2n)v_n \\ ev_n &= (n+1)v_{n+1} \\ fv_n &= (\lambda - n + 1)v_{n-1}. \end{aligned} \tag{5.76}$$

Clearly, the operators e and f are the creation and the annihilation operators according to existing in physics terminology while the vector v is being interpreted as the vacuum state vector. The question arises: how this vector is related to earlier introduced vector $\mid \alpha >$? Before providing the answer to this question we need, following Serre, to settle the related issue. In particular, we can either: a) assume that for all $n \geq 0$ the first of Eq.s(5.76) has solutions and all vectors v, v_1, v_2 , ..., are linearly independent or b) beginning from some $m+1 \geq 0$, all vectors v_n are zero, i.e. $v_m \neq 0$ but $v_{m+1} = 0$. The first option leads to the infinite dimensional representations associated with Kac-Moody affine algebras just mentioned. The second option leads to finite dimensional representations and to the requirement $\lambda = m$ with m being an integer. Following Serre, this observation can be exploited further thus leading us to crucial physical identifications. Serre observes that with respect to $n = 0$ Eq.s(5.76) possess a ("super")symmetry . That is the linear mappings

$$e^m : V^m \to V^{-m} \text{ and } f^m : V^{-m} \to V^m \tag{5.77}$$

are isomorphisms and the dimensionality of V^m and V^{-m} are the same. Serre provides an operator (the analog of Witten's F operator) $\theta = \exp(f)\exp(e)\exp(-f)$ such that $\theta \cdot f = -e \cdot \theta$, $\theta \cdot e = -\theta \cdot f$ and $\theta \cdot h = -h \cdot \theta$. In view of such operator, it is convenient to redefine h operator : $h \to \hat{h} = h - \lambda$. Then, for such redefined operator the vacuum state is just v. Since both L and $L^* = \Lambda$ commute with the supersymmetric Hamiltonian H and because of group endomorphism we conclude that the vacuum state $\mid \alpha >$ for H corresponds to the primitive state vector v.

Now we are ready to apply yet another isomorphism following Ginzburg [9],

ch-r 4[38]. To this purpose we make the following identification

$$e_m \to t_{m+1}\frac{\partial}{\partial t_m}\ ,\ f_m \to t_m\frac{\partial}{\partial t_{m+1}}\ ,\ h_m \to t_{m+1}\frac{\partial}{\partial t_{m+1}} - t_m\frac{\partial}{\partial t_m} \tag{5.78}$$

Such operators are acting on the vector space made of monomials of the type

$$v_k \to \mathcal{F}_n = \frac{n!}{n_1!n_2!\cdots n_k!}t_1^{n_1}\cdots t_k^{n_k} \tag{5.79}$$

where $n_1 + ... + n_k = n$. This result should be compared with Eq.(5.37). Eq.s (5.76) have now their analogs

$$\begin{aligned} h_m * \mathcal{F}_n(m) &= (n_{m+1} - n_m)\mathcal{F}_n(m) \\ e_m * \mathcal{F}_n(m) &= n_m\mathcal{F}_n(m+1) \\ f_m * \mathcal{F}_n(m) &= n_{m+1}\mathcal{F}_n(m-1) \end{aligned} \tag{5.80}$$

where, clearly, one should make the following consistent identifications: $m - 2n = n_{m+1} - n_m$, $n_m = n+1$ and $m - n + 1 = n_{m+1}$. Next, we define the total Hamiltonian: $h = \sum_{m=1}^{k} h_m$ and consider its action on the total wave function, e.g. see Eq.(5.37), $\sum\limits_{n=(n_1,\ldots,n_k)} \frac{n!}{n_1!n_2!\ldots n_k!}t_1^{n_1}\cdots t_k^{n_k}$. Upon redefining the Hamiltonian as before we obtain the ground state degeneracy equal to $p(k,n)$ in accord with Eq.(5.3.a). According to Witten [92], this degeneracy determines the Euler characteristic χ, e.g. see Eq.s(5.60),(5.73). Vergne [51] also obtained $p(k,n)$ using the standard procedure based on Atiyah-Singer-Hirzebruch index calculations. It involves calculation of traces of the Dirac operator as it is given by Eq.(5.73). Obtained results provide alternative explanation of the quantum mechanical nature of the multiparticle Veneziano (and Veneziano-like) amplitudes.

6 Instead of Discussion

Although this paper came out as rather long, in reality, much more remains to be done. In our previous work, Ref. [2], on Veneziano amplitudes we noticed connections with singularity theory, number theory, knot theory, dynamical systems theory. Present work adds to this list the theory of exactly integrable systems, K-theory, algebraic geometry, combinatorics, etc. These are mathematical aspects of the unfolding story. There are however physical aspects no less fascinating. For instance, in his 1982 paper [92] Witten had noticed remarkable connections between the sypersymmetry and the Lorentz invariance. This connection can be extended now based on the results of this work. Indeed,

[38]Unfortunately, the original sourse contains minor mistakes. These are easily correctable. The corrected results are given in the text.

pseudo-reflections used frequently in this work are isometries of the complex hyperbolic space [99]. We have discussed properties of real hyperbolic space earlier in connection with mathematical problems related to AdS-CFT correspondence [100] and dynamics of 2+1 gravity [101]. The connection between the Lorentz space-time and the real hyperbolic geometry is beautifully summarized in Thurston's book [102]. As results of our earlier work [100] indicate, the hyperbolic ball model of real hyperbolic space is quite adequate for description of meaningful physical models and the boundary of hyperbolic space plays a crucial role in such description. For instance, the infinitesimal variations at the boundary of the Poincare$'$ disc model -the simplest model of hyperbolic space H^2- naturally produce the Virasoro algebra (e.g. see our work, Ref.[100], Section 7, for details). Extension of the method producing this algebra to, say, H^3 is complicated by the Mostow rigidity theorem also discussed in our paper [100]. This theorem tells unequivocally that the Teichmüller space for the hyperbolic 3-manifolds without boundaries is just a point. That is all hyperbolic surfaces in hyperbolic spaces H^n, $n > 2$ are rigid (nonbendable). This restriction can be lifted in certain cases discussed in our earlier work. As it is demonstrated in Goldman's monograph [99], the real hyperbolic space is just part of the complex hyperbolic space which, not too surprisingly, can be modelled by the complex hyperbolic ball model. What is surprising, however, is the fact that the group of isometries of the boundary of such ball is the Heisenberg group. The method of coadjoint orbits discussed in Section 3.1. and the Todd transform, Eq.(5.58), designed by Khovanskii-Pukhlikov are indications of much tighter connections between classical and quantum mechanics. If, at least locally, our space is complex hyperbolic, then the isometries of this space generated by pseudo-reflection groups are inseparable from quantum mechanical description of reality. Moreover, the reader should not be left with impression that treatment of CFT can avoid use of these pseudo-reflection groups. The monograph by Kane [84] contains important references on earlier works by Kac and Kac and Petersen indicating profound importance of these groups for CFT as well. The intriguing problem still remains: if in the case of real hyperbolic space H^n, $n > 2$, the Mostow rigidity theorem forbids deformations of almost all real hyperbolic structures what could be said about the analog of such theorem for complex hyperbolic spaces and the rigidity of complex structures in such spaces? Although at this moment we are not yet aware of the detailed answer to this question, still several remarks are appropriate at this point. In particular, in the monograph [103] on quantum groups by Chari and Pressley on page 435 the deformation of $sl_2(\mathbf{C})$ is discussed and on page 439 it is stated that such deformation leads to failure of Kirillov-Kostant method of coadjoint orbits. As results of this work indicate, for the purposes of calculation of the Veneziano-like amplitudes, there is no need for such drastic measures. Moreover, the projectivised version of $SL_2(\mathbf{C})$ associated with isometries of H^3 is connected part of the Lorentz group O(3,1) which, in turn, is isomorphic to $PSL_2(\mathbf{C})$ as stated in our earlier work, Ref. [100] on AdS-CFT correspondence. Hence, it is impossible simultaneously to deform $SL_2(\mathbf{C})$, to keep the Lorentz invariance and to have quantum mechanics (the Heisenberg group) in its traditional form.

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Appendix A. Some results from the theory of Weyl-Coxeter reflection and pseudo-reflection groups

a) *The Weyl group*

As in Section 1.2., let V be a finite dimensional vector space endowed with a scalar product, Eq.(1.6), which is positive-definite symmetric bilinear form. For each nonzero $\alpha \in V$ [39]let r_α denote the orthogonal reflection in the hyperplane H_α through the origin perpendicular to α (i.e. set of hyperplanes H_α is in one-to one correspondence with set of $\alpha's$) so that for $v \in V$ we obtain

$$r_\alpha(v) = v - < v, \alpha^\vee > \alpha, \tag{A.1.}$$

where $\alpha^\vee = 2\alpha / < \alpha, \alpha >$ is the vector *dual* to α. Thus defined reflection is an orthogonal transformation in a sense that $< r_\alpha(v), r_\alpha(\mu) > = < \nu, \mu >$. In addition, $[r_\alpha(v)]^2 = 1 \ \forall \alpha, \nu$. Conversely, these two properties imply the transformation law, Eq.(A.1).

From these results it follows as well that for $v = \alpha$ we get $r_\alpha(\alpha) = -\alpha$ that is reflection in the hyperplane with change of vector orientation. If the set of vectors which belong to V is mutually orthogonal, then $r_\alpha(v) = v$ for $v \neq \alpha$ but, in general, the orthogonality is not required. Because of this, one introduces the *root system* Δ of vectors which span V. Such system is *crystallographic* if for each pair $\alpha, \beta \in \Delta$ one has

$$< \alpha^\vee, \beta > \in \mathbf{Z} \text{ and } r_\alpha(\beta) \in \Delta. \tag{A.2.}$$

Thus, each reflection r_α ($\alpha \in \Delta$) permutes Δ. Finite collection of such reflections forms a group W called *Weyl group of* Δ . The vectors $\alpha^\vee$ (for $\alpha \in \Delta$) form a root system $\Delta^\vee$ *dual* to Δ. Let $v \in \Delta$ be such that $< v,\alpha > \neq 0$ for each $\alpha \in \Delta$. Then, the set Δ^+ of roots $\alpha \in \Delta$ such that $< v,\alpha >> 0$ is called a system of *positive* roots of Δ. A root $\alpha \in \Delta^+$ is *simple* if it is not a sum of two elements from Δ^+.The number of simple roots coincides with the dimension of the vector space V and the root set Δ is made of disjoint union $\Delta = \Delta^+ \amalg \Delta^-$. The integral linear combinations of roots, i.e. $\sum_i m_i\alpha_i$ with $m_i's$ being integers, forms a root *lattice* $Q(\cdot)$ in V (that is free abelian group of rank $n = \dim V$). Clearly, the simple roots form a basis Σ of $Q(\cdot)$. Accordingly, $Q(\cdot^+)$ is made of combinations $\sum_i m_i\alpha_i$ with $m_i's$ being nonnegative integers.

In view of one-to -one correspondence between the set of hyperplanes $\cup_\alpha \mathrm{H}_\alpha$ and the set of roots Δ it is convenient sometimes to introduce **chambers** as connected components of the complement of $\cup_\alpha \mathrm{H}_\alpha$ in V. In the literature, Ref. [83], page 70, this complement is known also as the *Tits cone*.Accordingly, for a given chamber C_i its **walls** are made of hyperplanes H_α.The roots in Δ can therefore be characterized as those roots which are orthogonal to some wall of C_i and directed towards interior of this chamber. A **gallery** is a sequence (C_0, C_1,..., C_l) of chambers each of which is adjacent to and distinct from the next.

[39]In Section 1.2 we have used symbol u_i instead of α.

Let $w = r_{i_1}...r_{i_l}$ then, treating the Weyl group W as a chamber system, a gallery from 1 to w can be formally written as $(1, r_{i_1}, r_{i_1}r_{i_2}, ..., r_{i_1}...r_{i_l})$. If this gallery is of the shortest possible *lenght* $l(w)$ then one is saying that $r_{i_1}...r_{i_l}$ is *reduced decomposition* for the word w made of "letters" r_{i_j}. Let C_x and C_y be some distinct chambers which we shall call x and y for brevity. One can introduce the distance function $d(x, y)$ so that, for example, if $w = r_{i_1}...r_{i_l}$ is the reduced decomposition, then $d(x, y) = w$ if and only if there is a gallery of the type $r_{i_1}...r_{i_l}$ from x to y. If, for instance, $d(x, y) = r_i$, this means simply that x and y are distinct and $i-$adjacent. A **building** $\mathcal{B}$ is a chamber system having distance function $d(x, y)$ taking values in Weyl-Coxeter group W. Finally, an **apartment** in a building $\mathcal{B}$ is a subcomplex $\hat{\mathcal{B}}$ of $\mathcal{B}$ which is isomorphic to W. That is there is a bijection $\varphi : W \to \hat{\mathcal{B}}$ such that $\varphi(w)$ and $\varphi(w')$ are $i-$adjacent in $\hat{\mathcal{B}}$ if and only if w and w' are adjacent in W, e.g. see Ref.[104].

b) *The Coxeter group*

The Coxeter group is related to the Weyl group through the obviously looking type of relation between reflections

$$(r_\alpha r_\beta)^{m(\alpha,\beta)} = 1 \tag{A.3}$$

where, evidently, $m(\alpha, \alpha) = 1$ and $m(\alpha, \beta) \geq 2$ for $\alpha \neq \beta$. In particular, for *finite* Weyl groups $m(\alpha, \beta) \in \{2, 3, 4, 6\}$,Ref. [85], page39, while for the affine Weyl groups (to be discussed below) $m(\alpha, \beta) \in \{2, 3, 4, 6, \infty\}$, e.g. read Ref.[85], page 136, and Proposition A.1.below. Clearly, different refection groups will have different matrix $m(\alpha, \beta)$ and, clearly, the matrix $m(\alpha, \beta)$ is connected with bilinear form (Cartan matrix, see below) for the Weyl's group W [105].

As an example of use of the concept of building in the Weil group, consider the set of *fundamental weights* defined as follows. For the root basis Σ (or $\Sigma^\vee$) the set of fundamental weights $\mathcal{D}$ =$\{\omega_1, ..., \omega_n\}$ *with respect to* Σ is defined by the rule:

$$< \alpha_i^\vee, \omega_j >= \delta_{ij}. \tag{A.4}$$

The usefulness of such defined fundamental weights lies in the fact that they allow to introduce the concept of the *highest weight* λ *(* sometimes also known as dominant weight, [106] page 203). Thus defined λ can be presented as $\lambda = \sum_{i=1}^{d} a_i\omega_j$ with all $a_i \geq 0$. Sometimes it is convenient to relax the definition of fundamental weights to just weights by comparing Eq.s(A.2) and (A.4). That is $\beta's$ in Eq.(A.2) are just weights. Thus, for instance, we have Δ as building and subcomplex $\mathcal{D}$ of fundamental weights as an apartment complex.

To illustrate some of these concepts let us consider examples which are intuitively appealing and immediately relevant to the discussion in the main text. These are the root system B_d and C_d. They are made of vector set $\{u_1, ..., u_d\}$ constituting an orthonormal basis of the $d-$dimensional cube. The vectors u_i should not be necessarily of unit length, Ref. [84], page 27. It is important only that they all have the same length. For B$_d$ system one normally chooses, Ref. [84], page 30,

$$\Delta = \{\pm u_i \pm u_j \mid i \neq j\} \amalg \{\pm u_i\}. \tag{A.5}$$

In this case, the reflections corresponding to elements of Δ can be described by their effect on the set $\{u_1, ..., u_d\}$. Specifically,

$$r_{u_i - u_j} = \text{permutation which interchanges } u_i \text{ and } \mathrm{u}_j;$$
$$r_{u_i} = \text{sign change of } u_i;$$
$$r_{u_i + u_j} = \text{permutation which interchanges } u_i \text{ and } \mathrm{u}_j \text{ and changes their sign.}$$

The action of the Weyl group on Δ can be summarized by the following formula

$$W(\Delta) = (\mathbf{Z}/2\mathbf{Z})^d \trianglelefteq \Sigma_d \tag{A.6}$$

with $\trianglelefteq$ representing the semidirect product between the permutation group Σ_d and the dihedral group $(\mathbf{Z}/2\mathbf{Z})^d$ of sign changes both acting on $\{u_1, ..., u_d\}$.Thus defined product constitutes the full symmetry group of the $d-$cube, Ref. [84], page 31. The same symmetry information is contained in C_d root system defined by

$$\Delta = \{\pm u_i \pm u_j \mid i \neq j\} \amalg \{\pm 2u_i\} \tag{A.7}$$

Both systems possess the same root decomposition: $\Delta = \Delta^+ \amalg \Delta^-$ [84], page 37 . In particular, considering a square as an example we obtain the basis Σ_{B_2} of $Q(\cdot)$ as

$$\Sigma_{B_2} = \{u_1 - u_2, u_2\}. \tag{A.8a}$$

From here the dual basis is given by

$$\Sigma^{\vee}_{B_2} = \{u_1 - u_2, 2u_2\}. \tag{A.8b}$$

Using Eq.(A.4) we obtain the fundamental weights as $\omega_1 = u_1$ and $\omega_2 = \frac{1}{2}(u_1 + u_2)$ respectively which by design obey the orthogonality condition , Eq.(A.4). The Dynkin diagram, Ref. [84], page 122, for B_2 provides us with coefficients $a_1 = 1$ and a_2=2 obtained for the case when expansion $\lambda = \sum_{i=1}^{d} a_i\omega_j$ is relaxed to $\lambda = \sum_{i=1}^{d} a_i\beta_j$ as discussed above. In view of Eq.(A.8a) this produces at once: $\lambda_{B_2} = u_1 + u_2$. Analogously, for C_2 we obtain:

$$\Sigma_{C_2} = \{u_1 - u_2, 2u_2\}. \tag{A.9}$$

with coefficients $a_1 = 2$ and a_2=2 thus leading to $\lambda_{C_2} = 2\,(u_1 + u_2)$.

For the square, these results are intuitively obvious. Evidently, the $d-$ dimensional case can be treated accordingly. The physical significance of the highest weight should become obvious if one compares the Weyl-Coxeter reflection group algebra with that for the angular momentum familiar to physicists. In the last case, the highest weight means simply the largest value of the projection of the angular momentum onto z-axis. The raising operator will annihilate the wave vector for such quantum state while the lowering operator will produce all the eigenvalues lesser than the maximal value (up to the largest negative) and, naturally, all the eigenfunctions. The significance of the fundamental weights goes beyond this however. Indeed, suppose we can expand some root α_i according to the rule

$$\alpha_i = \sum_j m_{ij}\omega_j. \tag{A.10}$$

Then, substitution of such expansion into Eq.(A.2) and use of Eq.(A.4) produces

$$< \alpha_k^{\vee}, \alpha_i >= \sum_j m_{ij} < \alpha_k^{\vee}, \omega_j >= m_{ik} \tag{A.11}$$

The expression $< \alpha_k^{\vee}, \alpha_i >$ is known in the literature as Cartan matrix. It plays the central role in defining both finite and infinite dimensional semisimple Lie algebras [12]. According to Eq.s(A.4),(A.10),(A.11), the transpose of the Cartan matrix transforms the fundamental weights into the fundamental roots.

c) *The affine Weyl-Coxeter groups*

Physical significance of the affine Weyl-Coxeter reflection groups comes from the following proposition

Proposition A.1. *Let W be the Weyl group of any Kac-Moody algebra. Then W is a Coxeter group for which* $m(\alpha, \beta) \in \{2, 3, 4, 6, \infty\}$. *Any Coxeter group with such* $m(\alpha, \beta)$ *is crystallographic (e.g. see Eq.(A.2))*

The proof can be found in Ref.[105], pages 25-26.

To understand better the affine Weyl-Coxeter groups, following Coxeter, Ref.[22] , we would like to explain in simple terms the origin and the meaning of these groups. It is being hoped, that such discussion might significantly facilitate understanding of the results presented in the main text.

We begin with quadratic form

$$\Theta = \sum_{i,j} a_{ij} x_i x_j \tag{A.12}$$

with symmetric matrix $\|a_{ij}\|$ whose rank is ρ. Such form is said to be *positive definite* if it is positive for all values of $\mathbf{x} = \{x_1, ..., x_n\}$ ($n \geq \rho$ in general !) except zero. It is *positive semidefinite* if it is never negative but vanishes for some $x_i's$ not all zero. The form Θ is indefinite if it can be both positive for some $x_i's$ and negative for others.[40] If positive semidefinite form vanishes for some $x_i = z_i$ $(i = 1 - n)$, then

$$\sum_i z_i a_{ij} = 0, \ j = 1 - n. \tag{A.13}$$

For a given matrix $\|a_{ij}\|$ Eq.(A.13) can be considered as system of linear algebraic equations for $z_i's$. Let $\mathcal{N} = n - \rho$ be the *nullity* of the form Θ. Then, it is a simple matter to show that *every positive semidefinite connected* Θ *form is of nullity 1.* The form is connected if it cannot be presented as a sum of two forms involving separate sets of variables. The following two propositions play the key role in causing differences between the infinite affine Weyl-Coxeter (Kac-Moody) algebras and their finite counterparts discussed in subsections a) and b).

[40] For the purposes of comparison with existing mathematical physics literature [12] it is suficient to consider only positive and positive semidefinite forms.

Proposition A.2. *For any positive semidefinite connected Θ form there exist unique (up to multiplication by the common constant)* ***positive*** *numbers z_i satisfying Eq.(A.13).*

Proposition A.3. *If we modify a positive semidefinite connected Θ form by making one of the variables vanish, the obtained form becomes positive definite.*

Next, we consider the quadratic form Θ as the norm and the matrix a_{ij} as the metric tensor. Then, as usual, we have $\mathbf{x}\cdot\mathbf{x}=\Theta=|\mathbf{x}|^2$ and, in addition, $\mathbf{x}\cdot\mathbf{y}=\sum\limits_{i,j}a_{ij}x_iy_j\equiv\sum\limits_{i}x^iy_i=\sum\limits_{i}x_iy^i$ so that if vectors $\mathbf{x}$ and $\mathbf{y}$ are orthogonal we get $\sum\limits_{i,j}a_{ij}x_iy_j=0$ as required. Each vector $\mathbf{x}$ determines a point $(\mathbf{x})$ and a hyperplane $[\mathbf{x}]$ with respect to some reference point $\mathbf{0}$ chosen as origin. The distance l between a point $(\mathbf{x})$ and a hyperplane $[\mathbf{y}]$, measured along the perpendicular, is the projection of $\mathbf{x}$ along the direction of $\mathbf{y}$, i.e.

$$l=\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{y}|}. \tag{A.14}$$

Let now $(\mathbf{x}')$ be the image of $(\mathbf{x})$ by reflection in the hyperplane $[\mathbf{y}]$. Then, $\mathbf{x}-\mathbf{x}'$ is a vector parallel to $\mathbf{y}$ of magnitude $2l.$ Thus

$$\mathbf{x}'=\mathbf{x}-2\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{y}|}\mathbf{y} \tag{A.15}$$

in accord with Eq.(A.1). From here, equation for the reflecting hyperplane is just $\mathbf{x}\cdot\mathbf{y}=0$. Let the vector $\mathbf{y}$ be pre assigned then, taking into account Propositions A2 and A3 we conclude that for the nullity $\mathcal{N}=0$ the only solution possible is $\mathbf{x}=\mathbf{0}.$ That is to say in such case n reflecting hyperplanes have point $\mathbf{0}$ as the only common intersection point. Complement of these hyperplanes in $\mathbf{R}^n$ forms a chamber system discussed already in a). For $\mathcal{N}=1$ equation $\mathbf{x}\cdot\mathbf{y}=0$ may have many *nonnegative* solutions for $\mathbf{x}.$ Actually**,** such reflecting hyperplanes occur in a finite number of different directions. More accurately, such hyperplanes belong to *finite* number of families, each consisting of hyperplanes *parallel to each other*. If we choose a single representative from each family in a such a way that it passes through $\mathbf{0}$, then complement of such representatives is going to form a polyhedral cone as before. But now, in addition, we have a group of translations T for each representative of hyperplane family so that the total affine Weyl group W_{aff} is the semidirect product :$W_{aff}=T\trianglelefteq W.$ The fundamental region for W_{aff} is a simplex (to be precise, an open simplex, Bourbaki, Ref.[7], Chr.5, Proposition 10) called **alcove** bounded by $n+1$ hyperplanes (walls) n of which are reflecting hyperplanes passing through $\mathbf{0}$ while the remaining one serves to reflect $\mathbf{0}$ into another point $\mathbf{0}'$. If one connects $\mathbf{0}$ with $\mathbf{0}'$ and reflects this line in other hyperplanes one obtains a lattice. By analogy with solid state physics [36] one can construct a dual lattice (just like in a) and b) above) the fundamental cell of which is known in physics as the Brilluin zone. For the alcove the fundamental region of the dual lattice (the Brilluin zone) is the polytope having $\mathbf{0}$ for its centre of symmetry, i.e. zonotope[22].

d) *The pseudo-reflection groups*

Although the pseudo-reflection groups are also described by Bourbaki,Ref. [7], their geometrical (and potentially physical) meaning is beautifully explained in the book by McMullen [83]. In articular, all earlier presented reflection groups are isometries of Euclidean space. Their action preserves some quadratic form which is real. More generally, one can think of reflections in spherical and hyperbolic spaces. From this point of view earlier described polytopes (polyhedra) represent fundamental regions for respective isometry groups. Action of these groups on fundamental regions causes tesselation of these spaces (without gaps). The collection of spaces can be enlarged by considering reflections in complex n-dimensional space $\mathbf{C}^n$. In this case the Euclidean quadratic form is replaced by the positive definite Hermitian form. Since locally $\mathbf{CP}^n$ is the same as $\mathbf{C}^{n+1}$ and since $\mathbf{CP}^n$ is at the same time a symplectic manifold with well known symplectic two form Ω [43], this makes the pseudo-reflection groups (which leave Ω invariant) especially attractive for physical applications. This is indeed the case as the main text indicates. The pseudo-reflections are easily described. By analogy with Eq.(A.1) (or (A.15)) one writes

$$r_\alpha(v) = v + (\xi - 1) < v, \alpha^\vee > \alpha \tag{A.16}$$

where ξ is nontrivial solution of the cyclotomic equation $x^h = x$ and $\alpha^\vee = \alpha / < \alpha, \alpha >$ with $< x, y >$ being a positive definite Hermitian form satisfying as before $< r_\alpha(v), r_\alpha(\mu) > = < \nu, \mu >$ and α being an eigenvector such that $r_\alpha(\alpha) = \xi\alpha$[41]. In addition, $[r_\alpha(\alpha)]^k = \xi^k\alpha$ for $1 \leq k \leq h-1$. This follows from the fact that

$$[r_\alpha(\nu)]^k = \nu + (1 + \xi + \cdots + \xi^{k-1})(\xi - 1) < v, \alpha^\vee > \alpha \tag{A.17}$$

and taking into account that $(1 + \xi + \cdots + \xi^{k-1})(\xi - 1) = \xi^k - 1$.

Finally, the Weyl-Coxeter reflection groups considered earlier can be treated exactly as pseudo-reflection groups if one replaces a single Euclidean reflection by the so called Coxeter element [64] ω which is product of individual reflections belonging to the distinct roots of Δ. Hence, the Euclidean Weyl-Coxeter reflection groups can be considered as subset of pseudo-reflection groups so that useful information about these groups can be obtained from considering the same problems for the pseudo-reflection groups. It can be shown [64,84] that the Coxeter element ω has eigenvalues $\xi^{m_1}, ..., \xi^{m_l}$ with l being dimension of the vector space Δ while the exponents $m_1, ..., m_l$ are positive integers less than h and such that $\sum_{i=1}^{l}(h - m_i) = \sum_{i=1}^{l} m_i$. This result implies that the number $\sum_{i=1}^{l} m_i = N$ - the number of positive roots in the Weyl-Coxeter group is connected with the Coxeter number h via relation : $N = \frac{1}{2}lh$,Ref.[85],page79.

Appendix B. Analytical properties of the Veneziano and Veneziano-like 4- particle amplitudes.

[41] According to Bourbaki, Ref. [7], Chr5, paragraph 6, if ξ is an eigenvalue of psedo-reflection operator, then ξ^{-1} is also an eigenvalue with the same multiplicity.

a) *The Veneziano amplitudes*

Using Eq.(4.6) the four-particle Veneziano amplitude is given by

$$A(s,t,u) = \Gamma(-\alpha(s))\Gamma(-\alpha(t))\Gamma(-\alpha(u))[\sin\pi(-\alpha(s))+\sin\pi(-\alpha(t))+\sin\pi(-\alpha(u))] \tag{B.1}$$

To analyze the analytical properties of this amplitude we need to use the following known expansions

$$\sin\pi z = \pi z\prod_{k=1}^{\infty}(1-(\frac{z}{k}))(1+(\frac{z}{k})) \tag{B.2}$$

and

$$\frac{1}{\Gamma(z)} = ze^{-Cz}\prod_{k=1}^{\infty}(1+(\frac{z}{k}))e^{-\frac{z}{k}} \tag{B.3}$$

with C being Euler's constant

$$C = \lim_{n\to\infty}(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}-\ln n).$$

The above results when combined with the Veneziano condition, $\alpha(s)+\alpha(t)+\alpha(u)=-1$,Eq.(4.5), allow us to write (up to a constant factor) a typical singular portion of the Veneziano amplitude (the tachyons are to be considered separately):

$$\begin{aligned} A(s,t,u) &= \frac{1}{nlm(1-\frac{\alpha(s)}{n})}\frac{1}{(1-\frac{\alpha(t)}{m})}\frac{1}{(1-\frac{\alpha(u)}{l})}\times \\ &\quad [(1-\frac{\alpha(s)}{n})C(n)+(1-\frac{\alpha(t)}{m})C(m)+(1-\frac{\alpha(u)}{l})C(l)] \end{aligned} \tag{B.4}$$

where $C(n)$, etc. are known constants and m,n,l as some nonnegative integers. For actual use of this result the explicit form of these constants might be important. Looking at Eq.(B.2) we obtain,

$$C(n,\alpha) = \pi\alpha\frac{1}{(1-\frac{\alpha}{n})}\prod_{k=1}^{\infty}(1-(\frac{\alpha}{k}))(1+(\frac{\alpha}{k})). \tag{B.5}$$

where α can be $\alpha(s)$, etc. Clearly, this definition leads to further simplifications, e.g. to the manifestly symmetric form:

$$\begin{aligned} A(s,t,u) &= \frac{1}{nlm}[\frac{C(n,\alpha(s))}{(1-\frac{\alpha(t)}{m})}\frac{1}{(1-\frac{\alpha(u)}{l})} \\ &\quad +\frac{C(m,\alpha(t))}{(1-\frac{\alpha(s)}{n})}\frac{1}{(1-\frac{\alpha(u)}{l})} \\ &\quad +\frac{C(l,\alpha(u))}{(1-\frac{\alpha(s)}{n})}\frac{1}{(1-\frac{\alpha(t)}{m})}]. \end{aligned} \tag{B.4.a}$$

Consider now special case : $\alpha(s) = \alpha(t) = n$. In this case we obtain:

$$\begin{aligned} A(s &= t,u) = \frac{1}{n^2 m} \frac{1}{(1-\frac{\alpha(s)}{n})^2}[C(l,\alpha(u)) + 2C(n,\alpha(s))\frac{(1-\frac{\alpha(s)}{n})}{(1-\frac{\alpha(u)}{l})} \\ &= \frac{1}{n^2 m} \frac{1}{(1-\frac{\alpha(s)}{n})^2}[\sin \pi\alpha(u) + 2\sin \pi\alpha(s)]\frac{1}{(1-\frac{\alpha(u)}{l})} = 0 \qquad \text{(B.4b)} \end{aligned}$$

This result is in accord with that of Ref. [62] where it was obtained differently. The tachyonic case is rather easy to consider now. Indeed, using Eq.s(B.1)-(B.3) and taking into account the Veneziano condition, let us assume that, say, that $\alpha(s) = 0$ then, in view of symmetry of Eq.s.(B.4a),(B.4b), we need to let $\alpha(t) = 0$ as well to check if Eq(B.4b) holds. This leaves us with the option: $\alpha(u) = -1$.With such constraint we obtain (since $\Gamma(1) = 1$)

$$A(s,t,u) = \frac{\pi}{\alpha(t)} + \frac{\pi}{\alpha(s)} - \frac{\pi}{\alpha(t)} - \frac{\pi}{\alpha(s)} = 0$$

as required. Hence, indeed, even in the tachyonic case Eq.(B.4b) holds in accord with earlier results [62]. This means only that one cannot observe the tachyons in both channels simultaneously. But even to observe them in one channel is unphysical. Moreover, Eq.(B.4b) implies that only situations for which $\alpha(s) \neq \alpha(t) \neq \alpha(u)$ could be physically observable. By combining the Veneziano condition with such constraint leaves us with the following options:

a) $\alpha(s), \alpha(t) > 0, \alpha(u) < 0$;

b) $\alpha(s) > 0, \alpha(t), \alpha(u) < 0$ plus the rest of cyclically permuted inequalities.

This means that not only the tachyons of the type $\alpha(s) = 0$ (or $\alpha(t) = 0$, or $\alpha(u) = 0$) could be present but also those for which, for example, $\alpha(s) < 0$. This is so because according to results of standard open string theory [55] in 26 space-time dimensions $\alpha(s) = 1 + \frac{1}{2}s$. When $\alpha(s) = 0$ such convention produces the only one tachyon: $s = -2 = M^2$ and the whole spectrum (open string) is given by $M^2 = -2, 0, 2, ..., 2n$ where n is non negative integer. Incidentally, for the closed bosonic string under the same conditions the spectrum is known to be $M^2 = -8, 0, 8, ..., 8n$. No other masses are permitted. The requirements like those in a) and b) above produce additional complications. For instance, let $\alpha(s) = 1$,then consider the following option : $\alpha(t) = 3$ so that $\alpha(u) = -5$. This leads us to the tachyon mass : $M^2 = -12$. It is absent in the spectrum of both types of bosonic strings. The emerging apparent difficulty can actually be bypassed somehow due to the following chain of arguments. Consider, for example, the amplitude $V(s,t)$ and let both s and t be non tachyonic and, of course, $\alpha(s) \neq \alpha(t)$. Then, naturally, $\alpha(u) < 0$ *is* tachyonic but, when we use Eq.(B.4a), we notice at once that this creates no difficulty since $\alpha < 0$ condition simply will eliminate the resonance in the respective channels, e.g. if $V(s,t)$ will have poles for both s and t *then the same particle resonances will occur* in $V(s,u)$ and $V(t,u)$ channels so that, *except the case* $\alpha(s) = 0$ leading to the pole with mass $M^2 = -2$, no other tachyonic states will show up as resonances and, hence, they cannot be observed. At the same time, *effectively*, we have

only two types of resonances in the Veneziano amplitude which we would like to denote symbolically (up to permutation) as$\mathcal{V}_u(s)$ and $\mathcal{V}_u(t)$. The resonances for $\mathcal{V}_u(s) = V(s,t)+ V(s,u)$ and, accordingly, $\mathcal{V}_u(t) = V(t,s) + V(t,u)$. Such conclusion is valid only if one requires $V(s,t) = V(t,s)$. It is surely the case for the Veneziano amplitude, Eq.(B.4a). Accordingly, should Veneziano amplitude be free of tachyons (e.g. $\alpha(s) = 0$) it would be perfectly acceptable. Evidently, in the light of the results just obtained it can be effectively written as

$$A(s,t,u) = \mathcal{V}_u(s) + \mathcal{V}_u(t). \tag{B.5}$$

This result survives when instead of Veneziano we would like to use the Veneziano-like amplitudes obtained in the main text as demonstrated below.

b) *The Veneziano-like amplitudes*

Results of Section 4.4 allow us to write the following result for the 4- particle Veneziano-like amplitude, e.g. see Eq.s(4.46),(4.47),

$$A(s,t,u) = V(s,t) + V(s,u) + V(t,u), \tag{B.6}$$

where, for instance,

$$V(s,t) = (1 - \exp(i\frac{\pi}{N}(-\alpha(s))(1 - \exp(i\frac{\pi}{N}(-\alpha(t))B(-\frac{\alpha(s)}{N}, -\frac{\alpha(t)}{N}). \tag{B.7}$$

Although this result was obtained by the same analytic continuation as in the case of Veneziano amplitude, the resulting analytical properties of such Veneziano-like amplitude are markedly different. as we would like to demonstrate now.

We begin by noticing that in view of Eq.(4.41b) the Veneziano condition in its simplest form: $a + b + c = 1$, upon analytic continuation leads again to the requirement: $\alpha(s) + \alpha(t) + \alpha(u) = -1$ if we identify, for example, $\frac{<c_1>}{N}$ $= a_1$ with -$\alpha(s)$,etc. This naive identification leads to some difficulties however. Indeed, since physically we are interested in poles and zeros of gamma functions as results of previous subsection indicate, we expect our parameters a, b and c to be integers. This is possible only if absolute values of $< c_1 >, < c_2 >$ and $< c_3 >$ are greater or equal than N. By allowing these parameters to become greater than N we would formally violate the requirements of Corollary 2.11. by Griffiths, e.g. see Eq.(4.39) and discussion around it. Fortunately, the occurring difficulty can by resolved. For instance, one can postulate Eq.s (B.6),(B.7) as *defining equations* for the Veneziano-like amplitudes as it was done by Veneziano for Veneziano amplitudes. In this case one is confronted with the problem of finding the physical model producing such amplitudes. To facilitate search for such model it is reasonable to impose the same constraints as for the standard Veneziano amplitudes in the present case. Clearly, if we want to use the results of the main text, in addition to these constraints, the constraint coming from Corollary 2.11. should also be imposed. Corollary 2.11. formally forbids us from consideration of ratios $< c_i > /N$ whose absolute value is greater than one as it was discussed in Section 4.4.4. This fact creates no additional problems

however. This can be seen already on example of Eq.(B.2) for $\sin \pi x$. Indeed, consider the function

$$F(x) = \frac{1}{\sin \pi x}.$$

It will have the first order poles for $x = 0, \pm 1, \pm 2, ...$ If we define the bracket operator $< ... >$ by analogy with that defined in the main text, e.g. $0 << x > \leq 1 \ \forall x$, then to reproduce the poles of $F(x)$ it is sufficient to write

$$F(x) = \frac{1}{\sin \pi x} = \frac{1}{1 - < x >}. \tag{B.8a}$$

Clearly, the above result can be read as well from right to left, i.e. removal of brackets, is equivalent to unwrapping[42] S^1 into R^1. By looking at Eq.(B.3) for $\Gamma(z)$ and comparing it with Eq.(B.2) we notice that all singularities of $\Gamma(z)$ are exactly the same as those for $F(x)$. Hence, the same unwrapping is applicable for this case as well.

These observations lead us to the following set of prescriptions: a) use Eq.(4.45) in Eq.(4.46) in order to obtain the full Veneziano-like amplitude, b) remove brackets, c) perform analytic continuation to negative values of $c_i's$,d) identify $-c_i/N$ with $-\alpha(i)$,($i = s$, t or u). After this, let, for instance, $\alpha(s) = a + bs$ with both a and b being positive (or better, non negative) constants. Then, the tachyonic pole: $\alpha(s) = 0, n = 0$ is killed by the corresponding phase factor in Eq.(B.7). The mass spectrum is determined by a) the actual numerical values of constants a and b, b) by the phase factors and c) by the values of parameter N (even or odd).

For instance, the condition, Eq.(4.41b), leads to the requirement that the particle masses satisfying equation $\alpha(s) = 2l, l = 1, 2, ...$ cannot be observed since the emerging pole singularities are being killed by zeroes coming from the phase factor. In the case of 4-particle amplitude Eq.(4.61) should be used with $n = 1$ thus leading to constraints: $N \geq 3$ and $1 \leq m \leq 3$. If we choose $m = 3$ we obtain similar requirement forbidding particles with masses coming from equation $\alpha(s) = 3l, l = 1, 2, ...$

Such limitations are not too severe, however. Indeed, let us consider for a moment the existing bosonic string parameters associated with the Veneziano amplitude. For the open string the convention is $\alpha(s) = 1 + \frac{1}{2}s$ so that the tachyon state is determined by the condition: $\alpha(s) = 0$ producing $s = -2 = M^2$. If now $1 + \frac{1}{2}s = l$, we obtain: $s = 2(l - 1)$, $l = 1, 3, 5, ...$(for $m = 2$) or $l = 1, 2, 4, 5, ...$(for $m = 3$).Clearly, these results produce the mass spectrum of open bosonic string (without tachyons). If we want the graviton to be present in the spectrum we have to adjust the values of constants a and b. For instance, it is known [55] that for closed string the tachyon occurs at $s = -8 = M^2$. This result can be achieved either if we choose $\alpha(s) = 2 + \frac{1}{4}s$ or $\alpha(s) = 1 + \frac{1}{8}s$. To decide which of these two expressions fits better experimental data we recall that the massless graviton should have spin equal to two. If we want the graviton to be present in the spectrum we must select the first option. This is so because

[42] That is going to the universal covering space

of the following arguments. First, we have to take into account that for large s and fixed t the amplitude $V(s,t)$ can be approximated by [55], page 10,

$$V(s,t) \simeq \Gamma(-\alpha(t))(-\alpha(s))^{\alpha(t)} \tag{B.9}$$

while the Regge theory predicts [55], pages 3,4, that

$$V_J(s,t) = -\frac{g^2(-s)^J}{t - M_J^2} \simeq \frac{-g^2(-\alpha(s))^J}{\alpha(t) - J} \tag{B.10}$$

for the particle with spin J. This leaves us with the first option. Second, by selecting this option our task is not complete however since so far we have ignored the actual value of the Fermat parameter N. Such ignorance causes emergence of the fictitious tachyon coming from the equation $2 + \frac{1}{4}s = l$ for $l = 1$.This difficulty is easily resolvable if we take into account that the "Shapiro-Virasoro" condition, Eq.(4.41b), is reducible to the "Veneziano condition, Eq.(4.41a), in case if all $< c_i >$ in Eq.(4.41b) are even. At the same time, if N in Eq.(4.41a) is even, it can be brought to the form of Eq.(4.41b). Hence, in making identification of $-c_i/N$ with $-\alpha(i)$ we have to consider two options: a) N is odd, then $c_i/N = \alpha(i)$, b) N is even, $N = 2\hat{N}$, then $c_i/\hat{N} = \alpha(i)$.The fictitious tachyon is removed from the spectrum in case if we choose the option b). Clearly, after this, in complete analogy with the "open string" case we re obtain tachyon-free spectrum of the "closed" bosonic string.

To complete our investigation of the Veneziano-like amplitudes we still need some discussion related to Eq.s(B.9),(B.10). To this purpose, using the integral representation of Γ given by

$$\Gamma(x) = \int_0^\infty \frac{dt}{t} t^x \exp(-t)$$

and assuming that x is large and positive the leading term saddle point approximation is readily obtained so that we obtain the leading term

$$\Gamma(x) = A x^x \exp{(-x)}\,,$$

where A is some constant. Applying with some caution this result to

$$V(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s) - \alpha(t))} \equiv B(-\alpha(s), -\alpha(t))$$

we obtain Eq.(B.9)[43]. Although such arguments formally explain the origin of the Regge asymptotic law, Eq.(B.9), they do not illuminate the combinatorial origin of this result essential for its generalization. To correct this deficiency, following Hirzebruch and Zagier [90] let us consider the identity

$$\begin{aligned} \frac{1}{(1-tz_0)\cdots(1-tz_k)} &= (1 + tz_0 + (tz_0)^2 + ...) \cdots (1 + tz_n + (tz_n)^2 + ...) \\ &= \sum_{n=0}^{\infty} (\sum_{k_1+...+k_k=n} z_0^{k_0} \cdots z_k^{k_k}) t^n. \end{aligned} \tag{B.11}$$

[43]This result is more carefuly reobtained below.

When $z_0 = ... = z_n = 1$,the inner sum in the last expression yields the total number of monomials of the type $z_0^{k_0} \cdots z_n^{k_n}$ with $k_0 + ... + k_k = n$. The total number of such monomials is given by the binomial coefficient [79]

$$p(k,n) \equiv \left(\begin{array}{c} n+k \\ k \end{array} \right) = \frac{(n+1)(n+2)\cdots(n+k)}{k!}. \tag{B.12}$$

Hence, for this case Eq.(B11) is converted to useful expansion,

$$P(k,t) \equiv \frac{1}{(1-t)^{k+1}} = \sum_{n=0}^{\infty} p(k,n)t^n. \tag{B.13}$$

Taking into account the integral presentation of beta function, replacing $k+1$ by $\alpha(s)+1$ in Eq.(B.13) and using it in beta function representation of $V(s,t)$ produces after straightforward integration the following result

$$V(s,t) = -\sum_{n=0}^{\infty} p(\alpha(s),n) \frac{1}{\alpha(t)-n} \tag{B.14}$$

well known in string theory [55].The r.h.s. of Eq.(B.14) can be interpreted as the Laplace transform of the partition function, Eq.(B.13). Eq.(1.39) of Section 1, indicates that such interpretation is not merely formal. To see this, following Vergne [51] consider a region Δ_k of $\mathbf{R}^k$ consisting of all points $\nu = (t_1, t_2, ..., t_k)$, such that coordinates t_i of ν are non negative and satisfy the equation $t_1 + t_2 + ... + t_k \leq 1$. Clearly, such restriction is characteristic for the simplex in $\mathbf{R}^k$.Consider now the *dilated* simplex $n\Delta_k$ for some nonnegative integer n. The volume of $n\Delta_k$ is easily calculated and is known to be

$$\mathrm{vol}(n\Delta_k) = \frac{n^k}{k!}. \tag{B.15}$$

Next, let us consider points $\nu = (u_1, u_2, ..., u_k)$ with *integral* coordinates *inside* the dilated simplex $n\Delta_k$. The total number of points with integral coordinates inside $n\Delta_k$ is given by $p(k,n)$, Eq.(B.12), i.e.

$$p(k,n) = \left| n\Delta_k \cap \mathbf{Z}^k \right| = \frac{(n+1)(n+2)\cdots(n+k)}{k!}. \tag{B.16}$$

The function $p(k,n)$ happen to be the non negative integer. It arises naturally as the dimension of the quantum Hilbert space associated (through the coadjoint orbit method) with symplectic manifold of dimension $2k$ constructed by "inflating" Δ_k. Physically relevant details are provided in Section 5 while here we only use these observations to complete our discussion of the Regge-like result, Eq.(B.9).To this purpose using Eq.(B.14) and assuming that $\alpha(t) \to k^*$ we can approximate the amplitude $V(s,t)$ by

$$V(s,t) \simeq -\frac{p_{\alpha(s)}(k^*)}{\alpha(t)-k^*} \simeq -\frac{p_{\alpha(s)}(\alpha(t))}{\alpha(t)-k^*} \tag{B.17}$$

For large $\alpha(s)$[44] by combining Eq.s(B.16) and (B.17) we obtain

$$V(s,t) \simeq \frac{-1}{\alpha(t)-k^*}\frac{\alpha(s)^{k^*}}{k^*!} \tag{B.18}$$

In view of the footnote remark, and taking into account that $k^* \simeq \alpha(t)$, this result coincides with Eq.(B.9) as required. In addition, however, for large $k's$ it can be further rewritten as

$$V(s,t) \simeq \frac{-1}{\alpha(t)-k^*}(\frac{\alpha(s)}{k^*})^{k^*} \simeq \frac{-1}{\alpha(t)-k^*}(\frac{\alpha(s)}{\alpha(t)})^{\alpha(t)} \tag{B.19}$$

Obtained result is manifestly symmetric with respect to exchange $s \rightleftharpoons t$ in accord with earlier mentioned requirement $V(s,t) = V(t,s)$. Moreover, it explicitly demonstrates that the angular momentum of the graviton is indeed equal to two as required.

[44]Notice that the negative sign in front of $\alpha(s)$ was already taken into account

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